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Interpolation of several Banach spaces

by

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0. Introduction.

The theory of interpolation usually deals with interpolation of two topological vector spaces (Hilbert, Banach, quasi-Banach or even locally convex spaces). In the case of Banach spaces, we mention in particular the K- and J-methods (see e.g. Peetre [16] or Butzer-Berens [5], ch. III).

The purpose of this paper is to work out and apply a corresponding theory of interpolation of several (more than two) Banach spaces.

However, it turns out that on essential points in the theory the step from two to several spaces is not possible in general. The difficulties occur in connection with the so-called fundamental lemma. This lemma is needed in proving the equivalence of K- and J-methods. It says that, given any Banach couple $\vec{A} = \{A_0, A_1\}$, every $a \in \Sigma(\vec{A})$ in the closure of $\Delta(\vec{A})$ has the representation

$$a = \int_0^1 u(t) \frac{dt}{t}, \quad J(t, u(t)) \leq \gamma K(t, a),$$

where γ is a universal constant. For details we refer to Section 5. There we give in particular a counter-example (remark 5.1), showing that the analogue of the fundamental lemma fails for Banach $(n+1)$ -tuples $\vec{A} = \{A_0, A_1, \dots, A_n\}$ if $n > 1$. However, in the applications to follow (Section 10), these deficiencies will cause no trouble. In fact, in many concrete situations one can show directly that the analogue of the fundamental lemma holds true.

Let us mention a typical case to which our theory is applicable: Consider functions of two variables x_1, x_2 . For $1 < p < \infty$, $\vec{r} = (r_1, r_2)$, where r_1, r_2 are non-negative integers, let $S_p^{\vec{r}}L$ be the Banach space corresponding to the norm

$$\|f\|_{L_p} + \left\| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \right\|_{L_p} + \left\| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \right\|_{L_p} + \left\| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \right\|_{L_p}.$$

The theory of interpolation usually deals with interpolation of two topological vector spaces (Hilbert, Banach, quasi-Banach or even locally convex spaces). In the case of Banach spaces, we mention in particular the K - and J -methods (see e.g. Peetre [10] or Butzer-Buren [5], ch. III). The purpose of this paper is to work out and apply a corresponding theory of interpolation of several (more than two) Banach spaces.

However, it turns out that an essential point in the theory the step from two to several spaces is not possible in general. The difficulties occur in connection with the so-called *rearranged* norms. This norm is needed in proving the boundedness of K - and J -methods. It says that, given any Banach couple $\vec{X} = (X_0, X_1)$, every $x \in \vec{X}(\lambda)$ in the closure of $\vec{X}(\lambda)$ has the representation

$$x = \int_0^1 \phi(t) \, dt, \quad \|\phi(t)\|_{\vec{X}(\lambda)} \leq \lambda(t), \quad \phi(t) \geq 0.$$

where λ is a universal constant. For details we refer to Section 2. There we give in particular a counter-example (Theorem 2.10) showing that the closure of the fundamental norm holds for Banach couples $\vec{X} = (X_0, X_1)$ with $\lambda > 0$. However, for the construction of the K -method (Section 4), some restriction will result in general. At first, a very concrete restriction can be shown directly from the definition of the fundamental norm (Section 4).

Let us mention a typical case to which our theory is applicable. Consider functions of two variables $f(x, y)$ with $x, y \in \mathbb{R}^n$ and $f(x, y) \geq 0$. Let $\vec{X} = (X_0, X_1)$ be a Banach couple. Then, for any non-negative integer k , let

$$\vec{X}^k = (X_0^k, X_1^k) \quad \text{with} \quad X_i^k = \{f(x, y) : f(x, y) \geq 0, \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y) \, dx \, dy < \infty, \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y) \, dx \, dy = 1\}$$

Suppose that the vectors $\vec{r}^1$ and $\vec{r}^2$ are non-parallel. (Note in particular the case $\vec{r}^1 = (r_1, 0)$, $\vec{r}^2 = (0, r_2)$). Then by interpolation with respect to $\{L_p, S_p^{\vec{r}^1} L, S_p^{\vec{r}^2} L\}$ we obtain a space of Besov type $S_p^{\vec{r}^q} B$. Such spaces are introduced and studied by Nikolskij and his school. They can be characterized by means of Hölder conditions, involving the differences $f(x_1+h_1, x_2) - f(x_1, x_2)$, $f(x_1, x_2+h_2) - f(x_1, x_2)$, and the mixed difference $f(x_1+h_1, x_2+h_2) - f(x_1+h_1, x_2) - f(x_1, x_2+h_2) + f(x_1, x_2)$.

In many respects our paper overlaps with a paper by Yoshikawa [25]. The subject has also been treated by Kerzman [10], [11]. The first example of interpolation of several spaces appears in the famous paper by Foias and Lions [6].

The plan of the paper is as follows. In Section 1 we introduce the general concepts of interpolation functor and interpolation space. In Section 2 we define two important functionals, the K - and J -functionals, and in Section 3 a function norm $\Phi_{\vec{\theta}^q}$. We are then, in Section 4, ready for the definitions of the particular interpolation spaces, the K - and J -spaces, that the rest of the paper deals with. In Section 5 we study the connection between these spaces. Here we meet for the first time the above difficulties in passing from two to several spaces. In Section 6 we derive a few other properties of the K - and J -spaces. For example we find that if two or more of the spaces are equal, then there is a connection between interpolation functors of different orders. In Section 7 we prove the so-called power theorem, on which Section 8, interpolation of L_p -spaces, is based. In Section 9 we study the stability of K - and J -spaces. Here we also establish, in a general case, a connection between interpolation functors of different orders. In Section 10 we then apply the theory to some spaces of differentiable functions (of Besov- and Sobolev-type). The results

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are of the kind mentioned above. In an Appendix, finally, we discuss a generalization of the fundamental lemma. However this generalization goes in a direction which is of no value for the study of K - and J -spaces.

The subject of this paper was suggested to me by Professor Jaak Peetre. I thank him for great interest and valuable advice in my work. I also want to thank Dr. Jörgen Löfström, who has read the manuscript and given many valuable comments.

1. Banach tuples and interpolation functors.

In the following definitions we will use the terminology of categories and functors (cf. Peetre [23], where these concepts are used in connection with interpolation).

Definition 1.1. $\mathcal{E}_n$ is the category whose objects and morphisms are defined as follows:

The objects of $\mathcal{E}_n$ are all $(n+1)$ -tuples $\vec{A} = \{A_0, A_1, \dots, A_n\}$ of Banach spaces $A_0, A_1, \dots, A_n$, such that each A_i ($i = 0, 1, \dots, n$) can be linearly and continuously imbedded in a Hausdorff topological vector space $\mathcal{A}$ (which may vary with $\vec{A}$).

Let $\vec{A} = \{A_0, A_1, \dots, A_n\}$ and $\vec{B} = \{B_0, B_1, \dots, B_n\}$ be two such tuples, consisting of spaces imbedded in $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, by a morphism

$$T : \vec{A} \rightarrow \vec{B}$$

we mean a homomorphism (i.e. linear mapping) $T : \mathcal{A} \rightarrow \mathcal{B}$ such that $T|_{A_i}$ (the restriction of T to A_i) is a continuous homomorphism $A_i \rightarrow B_i$ ($i = 0, 1, \dots, n$). $\neq$

Thus $\mathcal{E}_0$ is the category of Banach spaces with morphisms defined by the continuous homomorphisms between such spaces.

If $n = 1$ we speak about Banach couples $\vec{A} = \{A_0, A_1\}$, if $n = 2$ about Banach triples $\vec{A} = \{A_0, A_1, A_2\}$, and generally about Banach $(n+1)$ -tuples $\vec{A} = \{A_0, A_1, \dots, A_n\}$.

The set of all subspaces of a vector space forms, in a natural way, a lattice. Thus we can speak of the sum (or linear hull)

$$\Sigma(\vec{A}) = A_0 + A_1 + \dots + A_n,$$

and intersection

$$\Delta(\vec{A}) = A_0 \cap A_1 \cap \dots \cap A_n$$

of $\vec{A}$. A norm on $\Sigma(\vec{A})$ is defined by

$$(1.1) \quad a \mapsto \|a\|_{\Sigma(\vec{A})} = \inf (\|a_0\|_{A_0} + \|a_1\|_{A_1} + \dots + \|a_n\|_{A_n}),$$

where the infimum is taken over all decompositions $a = a_0 + a_1 + \dots + a_n$, $a_i \in A_i$ ($i = 0, 1, \dots, n$). A norm on $\Delta(\vec{A})$ is given by

$$(1.2) \quad a \mapsto \|a\|_{\Delta(\vec{A})} = \max(\|a\|_{A_0}, \|a\|_{A_1}, \dots, \|a\|_{A_n}).$$

One readily verifies that provided with these norms $\Sigma(\vec{A})$ and $\Delta(\vec{A})$ are Banach spaces.

We recall the following definition: Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. By a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ we mean a rule which to every object A in $\mathcal{C}$ assigns an object $F(A)$ in $\mathcal{C}'$, and to every morphism T in $\mathcal{C}$ a morphism $F(T)$ in $\mathcal{C}'$, in such manner that:

$$\text{If } T : \vec{A} \rightarrow \vec{B}, \text{ then } F(T) : F(\vec{A}) \rightarrow F(\vec{B}),$$

$$F(T \circ S) = F(T) \circ F(S),$$

$$F(\text{id}_{\vec{C}}) = \text{id}_{F(\vec{C})}.$$

It is easily seen that Σ and Δ are functors $\mathcal{C}_n \rightarrow \mathcal{C}_0$. Here we let the morphism $T : \vec{A} \rightarrow \vec{B}$ be transformed into the restriction of T to $\Sigma(\vec{A})$ or $\Delta(\vec{A})$ respectively.

Definition 1.2. By an interpolation functor (or interpolation method) of order n , we mean a functor $F : \mathcal{C}_n \rightarrow \mathcal{C}_0$ such that

$$(i) \quad \Delta(\vec{A}) \subset F(\vec{A}) \subset \Sigma(\vec{A}) \text{ for any object } \vec{A} \text{ in } \mathcal{C}_n$$

(linear continuous imbeddings) and

$$(ii) \quad F(T) = T|_{F(\vec{A})} \text{ for any morphism } T : \vec{A} \rightarrow \vec{B} \text{ in } \mathcal{C}_n.$$

In other words we have the interpolation property:

$$T : \vec{A} \rightarrow \vec{B} \implies T : F(\vec{A}) \rightarrow F(\vec{B}),$$

valid for any two Banach tuples $\vec{A}$ and $\vec{B}$ in $\mathcal{C}_n$. Here and in the sequel we agree to write T instead of $F(T)$. By

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"abuse of language" we say that $F(\vec{A})$ is an interpolation space with respect to $\vec{A}$.

Now consider the operator norms:

$$M_i(T) = ||T||_{A_i, B_i} = \sup_{||a||_{A_i}=1} ||Ta||_{B_i} \quad (i = 0, 1, \dots, n),$$

$$M(T) = ||T||_{F(\vec{A}), F(\vec{B})} = \sup_{||a||_{F(\vec{A})}=1} ||Ta||_{F(\vec{B})}.$$

Set $\vec{M}(T) = (M_0(T), M_1(T), \dots, M_n(T))$.

Definition 1.3. Let f be a function $\bar{R}_+^{n+1} \rightarrow \bar{R}_+$ (non-negative real numbers). We say that F is an interpolation functor of type f , if there is a constant $C \geq 1$ such that

$$(1.3) \quad M(T) \leq C f(\vec{M}(T))$$

for any morphism T in $\mathcal{C}_n$. If F is of type f with $f(\vec{t}) = f(t_0, t_1, \dots, t_n) = \max(t_0, t_1, \dots, t_n)$, we say that F is bounded. If in addition $C = 1$ in (1.3), we say that F is exact.

One easily convince oneself that the functors Σ and Δ are exact interpolation functors.

The interpolation functors we are going to construct and study will be of type $t_0^{\theta_0} t_1^{\theta_1} \dots t_n^{\theta_n}$, where $\theta_i \geq 0$ ($i = 0, 1, \dots, n$) and $\sum_{i=0}^n \theta_i = 1$. Thus for the operator norms holds an inequality of convexity. In particular, in the case $n = 1$, we recognize the situation of for example the classical interpolation theorems of M. Riesz and Marcinkiewicz.

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properties of the function $f(x)$ defined by

the equation $f(x) = \frac{1}{2} (f(x-1) + f(x+1))$.

It is shown that $f(x)$ is a linear function of x and that

$$f(x) = \frac{1}{2} (f(x-1) + f(x+1)) \quad (1)$$

is satisfied for all x in the domain of f .

It is also shown that $f(x)$ is a linear function of x and that

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2. The K- and J-functionals.

In this section we define two functionals, the K- and J-functionals. By means of them, and certain "function norms", discussed in Section 3, we will define our interpolation spaces (Section 4).

First, however, we introduce some notation:

We often have to operate componentwise with tuples of real numbers, forming various new tuples. With $\vec{t} = (t_0, t_1, \dots, t_n) \in R^{n+1}$, $\vec{s} = (s_0, s_1, \dots, s_n) \in R^{n+1}$ we write (whenever the components are defined)

$$(2.1) \quad \vec{t} \cdot \vec{s} = (t_0 s_0, t_1 s_1, \dots, t_n s_n),$$

$$(2.2) \quad \vec{t}/\vec{s} = (t_0/s_0, t_1/s_1, \dots, t_n/s_n),$$

$$(2.3) \quad \vec{t}^{\vec{s}} = (t_0^{s_0}, t_1^{s_1}, \dots, t_n^{s_n}).$$

If $t_0 = t_1 = \dots = t_n = t$ or $s_0 = s_1 = \dots = s_n = s$ we omit the corresponding arrow, writing $t\vec{s}$, $t/\vec{s}$, $t^{\vec{s}}$ etc.

We define

$$\max(\vec{t}) = \max(t_0, t_1, \dots, t_n),$$

$$\min(\vec{t}) = \min(t_0, t_1, \dots, t_n).$$

If $t_i \geq s_i$ ($i = 0, 1, \dots, n$) we write $\vec{t} \geq \vec{s}$. Analogously, if $t_i > s_i$ ($i = 0, 1, \dots, n$) we write $\vec{t} > \vec{s}$. By R_+ ($\bar{R}_+$) we mean the positive (non negative) real numbers.

We will work within the topological algebra R^{n+1} (with multiplication defined by (2.1)). In particular, we consider R_+^{n+1} both as a subset of the vector space R^{n+1} and as a multiplicative group.

On the multiplicative group R_+^{n+1} we define an absolute value

$$(2.4) \quad |\vec{t}| = t_0 t_1 \dots t_n.$$

We shall often work with a specific affine hyperplane H^n in R^{n+1} . This hyperplane is defined by

$$\vec{t} \in H^n \quad \text{if and only if} \quad t_0 + t_1 + \dots + t_n = 1.$$

Its subsets $\underline{H_+^n} = H^n \cap R_+^{n+1}$ and $\underline{\bar{H}_+^n} = H^n \cap \bar{R}_+^{n+1}$ are of particular interest for us.

We will also work with the (projective) multiplicative topological group $\underline{P_+^n} = R_+^{n+1} / R_+ \vec{1}_n$ where $\vec{1}_n = (1, 1, \dots, 1) \in R_+^{n+1}$.

P_+^n is thus obtained from R_+^{n+1} by identification of all elements $\lambda \vec{t}$, $\lambda > 0$, with the multiplication inherited from R_+^{n+1} .

We admit ourselves to pass freely between R_+^{n+1} and P_+^n . Thus we use the same symbol $\vec{t}$ both to denote an element of R_+^{n+1} and the corresponding element of P_+^n .

A function f on R_+^{n+1} is said to be μ -homogeneous if

$$f(\lambda \vec{t}) = \lambda^\mu f(\vec{t}), \quad \lambda > 0.$$

By $\underline{\mathcal{H}_\mu} = \mathcal{H}_\mu(R_+^{n+1}, A)$ we mean the set of all μ -homogeneous Borel-measurable functions on R_+^{n+1} , with values in the Banach space A . In particular, the functions of $\mathcal{H}_0$ can also be considered as functions on P_+^n .

Let A be a Banach space. By tA , $t > 0$, we mean the same space with its original norm replaced by the norm $t || ||_A$.

If $\vec{A} = \{A_0, A_1, \dots, A_n\}$ is a Banach tuple, $\vec{t} = (t_0, t_1, \dots, t_n) \in R_+^{n+1}$, we write

$$\vec{t}\vec{A} = \{t_0 A_0, t_1 A_1, \dots, t_n A_n\}.$$

Definition 2.1. Let $\vec{A} = \{A_0, A_1, \dots, A_n\}$ and $\vec{t} = (t_0, t_1, \dots, t_n) \in R_+^{n+1}$.

For $a \in \Sigma(\vec{A})$ we define

$$(2.5) \quad K(\vec{t}, a; \vec{A}) = ||a||_{\Sigma(\vec{t}\vec{A})} = \inf (t_0 ||a_0||_{A_0} + t_1 ||a_1||_{A_1} + \dots + t_n ||a_n||_{A_n}).$$

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Here the infimum is taken over all decompositions

$$a = a_0 + a_1 + \dots + a_n, \quad a_i \in A_i \quad (i = 0, 1, \dots, n).$$

For $a \in \Delta(\vec{A})$ we define

$$(2.6) \quad J(t, a; A) = \|a\|_{\Delta(\vec{A})} = \max(t_0 \|a\|_{A_0}, t_1 \|a\|_{A_1}, \dots, t_n \|a\|_{A_n}).$$

Whenever possible we drop arguments in these expressions, writing thus $K(\vec{t}, a)$, $K(\vec{t})$, K etc.

Obviously $K(\vec{t}, a; \vec{A})$ is a norm on $\Sigma(\vec{A})$ for each $\vec{t} \in R_+^{n+1}$. This norm is equivalent to the norm (1.1). Similarly, $J(\vec{t}, a; \vec{A})$ gives a new (equivalent) norm on $\Delta(\vec{A})$.

Remark 2.1. Def. 2.1 differs if $n = 1$ from the usual one (cf. e.g. Peetre [16]) in the appearance of the variable t_0 . This will play no essential rôle in that case but leads to more symmetric calculations if $n > 1$.

Remark 2.2. One readily verifies that the function $K(\vec{t})$ is concave. In general it is not differentiable. However, it is always possible to find functionals with this property which are equivalent to K (prop. 7.1 below). An analogous statement can also be made about J .

The following proposition is an immediate consequence of def. 2.1. (cf. Peetre [16] or Butzer-Berens [5] for the case $n = 1$).

Proposition 2.1. We have the inequalities

$$(2.6) \quad K(\vec{t}, a) \leq \max(\vec{t}/\vec{s}) K(\vec{s}, a),$$

$$(2.7) \quad J(\vec{t}, a) \leq \max(\vec{t}/\vec{s}) J(\vec{s}, a),$$

$$(2.8) \quad K(\vec{t}, a) \leq \min(\vec{t}/\vec{s}) J(\vec{s}, a). \quad \neq$$

We leave the details to the reader.

3. The function norm $\Phi_{\vec{\theta}q}$.

The next step is to define a suitable "function norm". This is done in def. 3.1. We reproduce the considerations that lead to this definition.

Let E be a subspace of the real valued Borel-measurable functions. Generally, by a function norm on E we mean a functional, defined on E with values in $[0, \infty]$, which has the following properties:

$$(3.1) \quad \Phi[f] = 0 \quad \text{if and only if} \quad f = 0 \quad \text{a.e.,}$$

$$(3.2) \quad \Phi[f] < \infty \quad \text{implies} \quad |f| < \infty \quad \text{a.e.,}$$

$$(3.3) \quad \Phi[cf] = |c| \Phi[f],$$

$$(3.4) \quad |f| \leq \sum_{i=0}^{\infty} |f_i| \quad \text{a.e. implies} \quad \Phi[f] \leq \sum_{i=0}^{\infty} \Phi[f_i].$$

Our intention is to define interpolation functors of type $|\vec{s}^{\vec{\theta}}|$, where $\vec{\theta} \in \bar{H}_+^n$. This will be done by applying a suitably chosen function norm Φ on $\mathcal{H}_1$ to the K - and J -functionals (see def. 4.1 and 4.2 below). For example, in the K -case we form the space of all $a \in \Sigma(\vec{A})$ for which "the norm" $\Phi[K(\vec{t}, a; \vec{A})] < \infty$.

Now, let $T : \vec{A} \rightarrow \vec{B}$ and $\vec{M} = \vec{M}(T)$ be as in def. 1.3. Then

$$K(\vec{t}, T a; \vec{B}) \leq K(\vec{M} \vec{t}, a; \vec{A}).$$

We aim at an inequality

$$\Phi[K(\vec{t}, T a; \vec{B})] \leq |\vec{M}^{\vec{\theta}}| \Phi[K(\vec{t}, a; \vec{A})].$$

This is ensured by the following assumption on Φ (acting on the variable $\vec{t}$):

$$(3.5) \quad \Phi[f(\vec{s} \vec{t})] \leq |\vec{s}^{\vec{\theta}}| \Phi[f(\vec{t})], \quad f \in \mathcal{H}_1.$$

(A function norm with this property is said to be of genus $|\vec{s}^{\vec{\theta}}|$, cf. Peetre [16]).

But if Φ obeys (3.5), then also

$$(3.5') \quad \Phi[f(\vec{t})] \leq |\vec{s}^{-\vec{\theta}}| \Phi[f(\vec{s}\vec{t})].$$

Thus, in fact, (3.5) is equivalent to

$$(3.6) \quad \Phi[f(\vec{s}\vec{t})] = |\vec{s}^{\vec{\theta}}| \Phi[f(\vec{t})].$$

We now associate to Φ a function norm Φ_0 on $\mathcal{H}_0$ by (cf. Peetre [16])

$$(3.7) \quad \Phi_0[g(\vec{t})] = \Phi[|\vec{t}^{\vec{\theta}}| g(\vec{t})], \quad g \in \mathcal{H}_0.$$

Then, in view of (3.6), Φ_0 has the invariance property

$$(3.8) \quad \Phi_0[g(\vec{s}\vec{t})] = \Phi_0[g(\vec{t})], \quad g \in \mathcal{H}_0.$$

But the functions of $\mathcal{H}_0$ may be considered as functions on the locally compact group P_+^n , defined in Section 2. Let $\omega = \omega(\vec{t})$ be a Haar measure on P_+^n , i.e. a positive measure on P_+^n with the invariance property

$$(3.9) \quad \int_{P_+^n} g(\vec{s} \cdot \vec{t}) \omega(\vec{t}) = \int_{P_+^n} g(\vec{t}) \omega(\vec{t}).$$

Here $\vec{s} \in P_+^n$ and g is any function on P_+^n , integrable with respect to ω . This measure ω is unique apart from a positive factor of proportionality. Thus ω will be fixed by the following normalization condition:

$$(3.10) \quad \int_{P_+^n} |\vec{t}|^{-1/n+1} \min(\vec{t}) \omega(\vec{t}) = (n+1)^{n+1}.$$

(Concerning integration on locally compact groups and Haar measure, we refer to e.g. Bourbaki [4]).

Reasonable definitions of Φ_0 , making it satisfy (3.1)-(3.4) and (3.8) are then expressed by

$$\Phi_0[g(\vec{t})] = \left(\int |g(t)|^q \omega(t) \right)^{1/q}, \quad 1 \leq q \leq \infty.$$

Taking into account (3.7) we are lead to

Definition 3.1. Let $1 \leq q \leq \infty$, $\vec{\theta} \in \bar{H}_+^n$ and let ω be the Haar measure on P_+^n normalized by (3.10). We define a function norm on $\mathcal{R}_1$ by

$$(3.11) \quad \Phi_{\vec{\theta}q}[f] = \left(\int_{P_+^n} |\vec{t}^{-\vec{\theta}}| |f(\vec{t})|^q \omega(\vec{t}) \right)^{1/q}$$

(with the usual interpretation if $q = \infty$).

To find the value of (3.11) in concrete cases we need to know $\omega(\vec{t})$ explicitly. We will find (prop. 3.1) that $\omega(\vec{t})$ is determined by an n -form on R_+^{n+1} . The integrations will be done over surfaces belonging to the following class:

Definition 3.2. Let Π be a subset of R_+^{n+1} such that the mapping

$$\Pi \ni \vec{t} \mapsto \vec{t} \in P_+^n$$

is a homeomorphism. We say that Π is a P-surface if it is composed by finitely many C^1 m -manifolds, $m \leq n$. On the n -manifold components an orientation is fixed by means of the unit normal vector field pointing into the set $\lambda\Pi$, $\lambda > 1$.

(If we omit the differentiability condition, Π is said to be a "fundamental domain", cf. Bourbaki [4], VII, 2, 10).

As examples of P-surfaces we mention H_+^n , $|\vec{t}| = 1$, $\max(\vec{t}) = 1$, $\min(\vec{t}) = 1$ and $t_j = 1$ ($j = 0, 1, \dots, n$).

Proposition 3.1. The Haar measure ω on P_+^n is determined by the n -form on R_+^{n+1}

$$(3.12) \quad \sum_{j=0}^n (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \overset{\vee}{\frac{dt_j}{t_j}} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

More precisely we have for any P-surface Π that

$$(3.13) \quad \int_{P_+^n} g(\vec{t}) \omega(\vec{t}) = \int_{\Pi} g(\vec{t}) \sum_{j=0}^n (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \overset{\vee}{\frac{dt_j}{t_j}} \wedge \dots \wedge \frac{dt_n}{t_n}, \quad g \in \mathcal{R}_0.$$

(Here the symbol $\vee$ indicates a missing factor).

Proof: We will use the forms

$$\lambda(\vec{t}) = \frac{dt_0}{t_0} \wedge \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n},$$

$$\bar{\omega}(\vec{t}) = \sum_{j=0}^n (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_j}{t_j} \wedge \dots \wedge \frac{dt_n}{t_n},$$

$$\frac{d|\vec{t}|}{|\vec{t}|} = \sum_{j=0}^n \frac{dt_j}{t_j}.$$

(Once the proposition is proved, we will not distinguish between ω and $\bar{\omega}$).

The proof is based on the relation

$$(3.14) \quad \lambda(\vec{t}) = \frac{1}{n+1} \frac{d|\vec{t}|}{|\vec{t}|} \wedge \bar{\omega}(\vec{t}).$$

Let $\phi \geq 0$, $\int \phi(r) \frac{dr}{r} = n+1$. With $g \in \mathcal{H}_0$, we define

$$f(\vec{t}) = \phi(|\vec{t}|)g(\vec{t}).$$

Let Π_i ($i = 0, 1, \dots, q$) be the n -manifold components of Π .

Then, by means of (3.14),

$$(3.15) \quad \int_{R_+^{n+1}} f(\vec{t})\lambda(\vec{t}) = \sum_{i=0}^q \int_{\Pi_i} g(\vec{t}) \left(\frac{1}{n+1} \int_{R_+^{\vec{t}}} \phi(|\vec{t}|) \frac{d|\vec{t}|}{|\vec{t}|} \right) \bar{\omega}(\vec{t}).$$

But here, by definition of ϕ , the interior integral on the right-hand side equals 1. Thus

$$(3.16) \quad \int_{R_+^{n+1}} f(\vec{t})\lambda(\vec{t}) = \int_{\Pi} g(\vec{t})\bar{\omega}(\vec{t}).$$

This proves the independence of Π in (3.13).

We now apply the same calculation to $f(\vec{st}) = \phi(|\vec{st}|)g(\vec{st})$, $\vec{s} \in R_+^{n+1}$. Then the interior integral on the right-hand side in (3.15) still equals 1, and we obtain

$$(3.17) \quad \int_{R_+^{n+1}} f(\vec{st})\lambda(\vec{t}) = \int_{\Pi} g(\vec{st})\bar{\omega}(\vec{t}).$$

Since $\lambda(\vec{t})$ is a Haar measure on R_+^{n+1} , the left-hand sides of

(3.16) and (3.17) are equal. Thus we have proved that for any P-surface Π , under the identification of $\mathcal{H}_0$ and the functions on P_+^n , the functional

$$g \longmapsto \int_{\Pi} g(t) \bar{\omega}(t)$$

defines a Haar integral on P_+^n .

What remains is the normalization condition (3.10). By integration over the P-surface $\min(\vec{t}) = 1$, we find

$$\int_{\min(\vec{t})=1} |\vec{t}|^{-1/(n+1)} \min(\vec{t}) \bar{\omega}(\vec{t}) = \sum_{j=0}^n \int_{\min(\vec{t})=t_j=1} |\vec{t}|^{-1/n+1} (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_j}{t_j} \wedge \dots \wedge \frac{dt_n}{t_n} = (n+1)^{n+1}$$

This concludes the proof. $\#$

Remark 3.1. In our simple situation we have preferred to give a direct proof of prop. 3.1. In the perspective of the general theory of Haar integrals, it can be interpreted in the following way:

i) The Haar measure property. Let $U_n = \{\vec{t} \in R_+^{n+1} \mid |\vec{t}| = 1\}$ (which is a subgroup of R_+^{n+1}). There is a unique decomposition $\vec{t} = \lambda \vec{u} = (\lambda \vec{1}_n) u$, with $\lambda = |\vec{t}|^{1/(n+1)}$, $\vec{u} \in U_n$. Hence $R_+^{n+1} = (R_+ \vec{1}_n) U_n$ (direct product). From (3.14) then follows that

$$\bar{\omega}(\vec{t}) = \lambda(\vec{t}) / \frac{d|\vec{t}|^{1/n+1}}{|\vec{t}|^{1/n+1}}$$

is a Haar measure on U_n (Bourbaki [4], VII, 2,7 prop. 10).

But by definition $P_+^n = R_+^{n+1} / R_+ \vec{1}_n$. Thus P_+^n and U_n are isomorphic. In view of this isomorphism, (3.12) defines a Haar measure on P_+^n .

ii) The independence of P-surfaces. This is the content, in our particular case, of Bourbaki [4], VII, 2,10 th. 4. $\#$

In the sequel we will, somewhat improperly, use the letter ω to denote both the Haar measure on P_+^n and (the measure associated with) the n -form (3.12) on R_+^{n+1} . Thus we write

$$\omega(\vec{t}) = \sum_{j=0}^n (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_j}{t_j} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

For the terms we will write

$$\omega_j(\vec{t}) = (-1)^j \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_j}{t_j} \wedge \dots \wedge \frac{dt_n}{t_n} \quad (j = 0, 1, \dots, n)$$

Obviously, on the surfaces $t_j = 1$ we have $\omega = \omega_j$ ($j = 0, 1, \dots, n$).

Remark 3.2. In the case $n = 1$ we get, choosing for Π the surface $t_0 = 1$,

$$\Phi_{\vec{\theta}q} [f] = \left(\int_0^\infty |t_1^{-\theta_1} f(1, t_1)|^q \frac{dt_1}{t_1} \right)^{1/q}.$$

This is the customary expression in that case, cf. e.g. Peetre [16]. Here $\Phi_{\vec{\theta}q}$ is usually denoted $\Phi_{\theta_1 q}$. Similarly we get

for general n that

$$\Phi_{\vec{\theta}q} [f] = \left(\int_{R_+^n} |t_1^{-\theta_1} \dots t_n^{-\theta_n} f(1, t_1, \dots, t_n)|^q \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n} \right)^{1/q}.$$

In the sequel we will often refer to the following relations:

$$(3.19) \quad \Phi_{\vec{\theta}q} [f(\vec{s}\vec{t})] = |\vec{s}^{\vec{\theta}}| \Phi_{\vec{\theta}q} [f(\vec{t})] \quad \text{if } 1 \leq q \leq \infty, \vec{\theta} \in \bar{H}_+^n,$$

$$(3.20) \quad \Phi_{\vec{\theta}q} [\min(\vec{t})] < \infty \\ \text{if } 1 \leq q \leq \infty, \vec{\theta} \in H_+^n \text{ and/or } q = \infty, \vec{\theta} \in \bar{H}_+^n,$$

$$(3.20') \quad \left(\int (|\vec{t}^{\vec{\theta}}| \min(1/\vec{t}))^q \omega(\vec{t}) \right)^{1/q} < \infty \\ \text{if } 1 \leq q \leq \infty, \vec{\theta} \in H_+^n \text{ and/or } q = \infty, \vec{\theta} \in \bar{H}_+^n.$$

Of these, (3.19) is just a reproduction of (3.6) with $\Phi = \Phi_{\vec{\theta}q}$.

By a change of variables it is readily seen that (3.20) and

(3.20') are equivalent. To verify (3.20), let Π be the P -surface $\min(\vec{t}) = 1$. Then, if $1 \leq q < \infty, \vec{\theta} \in H_+^n$,

$$\int_{\Pi} (|\vec{t}^{-\vec{\theta}}|_{\min(\vec{t})})^q \omega(\vec{t}) = \sum_{j=0}^n \int_{\min(\vec{t})=t_j=1} |\vec{t}^{-\vec{\theta}}|^q \omega_j(\vec{t}).$$

But here all the integrals on the right-hand side are finite.

If $q = \infty$, $\vec{\theta} \in \bar{H}_+^n$, the verification of (3.20) is immediate.

We conclude this section by stating some facts, that will be used in the proof of the reiteration theorem, th. 9.1, below.

Let ω denote the functional

$$\omega : g \mapsto \int g(\vec{t}) \omega(\vec{t}), \quad g \in \mathcal{H}_0.$$

Set

$$(\vec{s}g) : \vec{t} \mapsto g(\vec{s}\vec{t}).$$

Then the invariance property (3.9) can be reformulated as

$$(3.9') \quad \omega(\vec{s}g) = \omega(g).$$

Now let α be a topological automorphism on P_+^n , i.e.

$\alpha : P_+^n \rightarrow P_+^n$ is continuous, bijective and $\alpha(\vec{s}\vec{t}) = \alpha(\vec{s})\alpha(\vec{t})$ for all $\vec{s}, \vec{t} \in P_+^n$. Consider the functional

$$\omega' : g \mapsto \int g(\alpha(\vec{t})) \omega(\vec{t}), \quad g \in \mathcal{H}_0.$$

Let $\vec{s} = \alpha(\vec{u})$ (α is bijective!). Then

$$\begin{aligned} \omega'(\vec{s}g) &= \int (\vec{s}g)(\alpha(\vec{t})) \omega(\vec{t}) = \int g(\alpha(\vec{u})\alpha(\vec{t})) \omega(\vec{t}) = \\ &= \int g(\alpha(\vec{u}\vec{t})) \omega(\vec{t}) = \int g(\alpha(\vec{t})) \omega(\vec{t}) = \omega'(g). \end{aligned}$$

Hence ω' obeys (3.9'). The uniqueness of Haar measure then yields that for some constant $D(\alpha) > 0$

$$(3.21) \quad D(\alpha) \int g(\alpha(\vec{t})) \omega(\vec{t}) = \int g(\vec{t}) \omega(\vec{t}).$$

($D(\alpha)$ is the "modulus of α ", cf. Bourbaki [4], VII, 1.4).

Proposition 3.2. α is an automorphism on P_+^n if and only if

$$(3.22) \quad \alpha : P_+^n \ni \vec{t} \mapsto (|\vec{t}^{c\vec{\theta}^0}|, |\vec{t}^{c\vec{\theta}^1}|, \dots, |\vec{t}^{c\vec{\theta}^n}|) \in P_+^n,$$

where $c \neq 0$ and $\vec{\theta}^i \in H^n$ ($i = 0, 1, \dots, n$) are linearly independent in R^{n+1} .

If α is an automorphism on P_+^n , $\vec{\lambda} \in \bar{H}_+^n$, $\vec{\theta} = \sum_{i=0}^n \lambda_i \vec{\theta}^i$, then there exists a constant $D(\alpha) > 0$ such that

$$(3.23) \quad \Phi_{\vec{\lambda}q} [f(\vec{t})] = (D(\alpha) \int |\vec{t}^{-c\vec{\theta}} f(\alpha(\vec{t}))|^{q\omega(\vec{t})})^{1/q}.$$

If in particular $c = 1$, $\vec{\theta}^i \in \bar{H}_+^n$ ($i = 0, 1, \dots, n$), then $\vec{\theta} \in \bar{H}_+^n$ and

$$(3.24) \quad \Phi_{\vec{\lambda}q} [f(\vec{t})] = D(\alpha)^{1/q} \Phi_{\vec{\theta}q} [f(\alpha(\vec{t}))].$$

Proof: If α obeys (3.22), then α is an automorphism on P_+^n , and (3.23), (3.24) are immediate consequences of (3.21). The necessity of (3.22) (which in fact we will have no need for in what follows) is readily verified by means of elementary methods of linear algebra, using the isomorphism

$$R^{n+1} \ni \vec{t} \mapsto 2^{\vec{t}} \in R_+^{n+1}.$$

We omit the details. $\#$

Remark 3.3. A closer calculation yields that $D(\alpha) = c^n \det \theta$, where θ is the matrix with rows $\vec{\theta}^i$ ($i = 0, 1, \dots, n$).

4. The K- and J-spaces.

Proceeding as in the case $n = 1$, we now define our interpolation spaces by means of the K- and J-functionals and the function norm $\Phi_{\vec{\theta}q}$. The proofs of this section are entirely parallel to those in the case $n = 1$ (cf. Peetre [16], Ch. III, IV, or Butzer-Berens [5], Sec. 3.2). We admit ourselves to be somewhat concise.

Definition 4.1. Let $1 \leq q \leq \infty$, $\vec{\theta} \in H_+^n$ and/or $q = \infty$, $\vec{\theta} \in \bar{H}_+^n$. By $\vec{A}_{\vec{\theta}q;K}$ we mean the set of all $a \in \Sigma(\vec{A})$ for which

$$(4.1) \quad \Phi_{\vec{\theta}q} [K(\vec{t}, a; \vec{A})] < \infty.$$

Remark 4.1. In the case $n = 1$, $\vec{\theta} = (1-\theta, \theta)$, the space $\vec{A}_{\vec{\theta}q;K}$ is usually denoted $A_{\theta q;K}$.

Theorem 4.1. $\vec{A}_{\vec{\theta}q;K}$ is a Banach space under the norm

$$(4.2) \quad a \mapsto \|a\|_{\vec{A}_{\vec{\theta}q;K}} = \Phi_{\vec{\theta}q} [K(\vec{t}, a; \vec{A})].$$

Furthermore

$$(4.3) \quad K(\vec{t}, a; \vec{A}) \leq C |\vec{t}^{\vec{\theta}}| \|a\|_{\vec{A}_{\vec{\theta}q;K}} \quad \text{if } a \in \vec{A}_{\vec{\theta}q;K},$$

$$(4.4) \quad \|a\|_{\vec{A}_{\vec{\theta}q;K}} \leq C |\vec{t}^{-\vec{\theta}}| J(\vec{t}, a; \vec{A}) \quad \text{if } a \in \Delta(\vec{A}).$$

Hence

$$(4.5) \quad \Delta(\vec{A}) \subset \vec{A}_{\vec{\theta}q;K} \subset \Sigma(\vec{A}).$$

Proof: That (4.2) defines a norm is obvious. To prove (4.3), we rewrite (2.7) as

$$\min(\vec{s}/\vec{t}) K(\vec{t}, a) \leq K(\vec{s}, a).$$

Application of $\Phi_{\vec{\theta}q}$, acting on $\vec{s}$, gives

$$\Phi_{\vec{\theta}q} [\min(\vec{s}/\vec{t}) K(\vec{t}, a)] \leq \Phi_{\vec{\theta}q} [K(\vec{s}, a)].$$

Now (4.3) follows from (3.19), (3.20). In a similar manner, (4.4) is a consequence of (2.9). What remains is the completeness. This is proved, word by word, as in the case $n = 1$ (see the references given above). We omit the details. $\#$

That $\vec{A}_{\vec{\theta}q;K}$ is indeed an interpolation space is guaranteed by

Theorem 4.2. The functor

$$(4.6) \quad \mathcal{C}_n \ni \vec{A} \mapsto \vec{A}_{\vec{\theta}q;K} \in \mathcal{C}_0$$

is an interpolation functor of type $|\vec{\theta}|$.

Proof: Condition (i) in the definition of interpolation functors (def. 1.2) is already verified in view of (4.5).

Next, consider a map $T : \vec{A} \rightarrow \vec{B}$. Let $\vec{M} = \vec{M}(T)$ be as in def. 1.3. Then

$$K(\vec{t}, Ta; \vec{B}) \leq K(\vec{M}\vec{t}, a; \vec{A}).$$

Thus, by (3.17),

$$\|Ta\|_{\vec{B}_{\vec{\theta}q;K}} \leq |\vec{M}\vec{\theta}| \|a\|_{\vec{A}_{\vec{\theta}q;K}}.$$

This is the desired inequality. (Indeed, it was this argument that lead to the definition of $\Phi_{\vec{\theta}q}$ in Section 3). $\#$

We refer to the functor (4.6) as the K-functor (or K-method).

Definition 4.2. Let $1 \leq q \leq \infty$, $\vec{\theta} \in H_+^n$ and/or $q = 1$, $\vec{\theta} \in \bar{H}_+^n$. By $\vec{A}_{\vec{\theta}q;J}$ we mean the set of all $a \in \Sigma(\vec{A})$, for which

there exists a function $u \in \mathcal{H}_0(\Delta(\vec{A}))$, such that

$$(4.7) \quad a = \int u(\vec{t})\omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}),$$

$$(4.8) \quad \Phi_{\vec{\theta}q} [J(\vec{t}, u(\vec{t}); \vec{A})] < \infty.$$

(Note that $J(\vec{t}, u(\vec{t}); \vec{A}) \in \mathcal{H}_1$ whenever $u \in \mathcal{H}_0$).

Remark 4.2. In the case $n = 1$, $\vec{\theta} = (1-\theta, \theta)$, the space

$\vec{A}_{\vec{\theta}q;J}$ is usually denoted $\vec{A}_{\theta q;J}$.

Remark 4.3. Actually, the absolute convergence of (4.7) is a consequence of (4.8). Thus we need only require that (4.7) holds in some weaker sense, e.g. as a generalized integral. To see this, let K be a compact subset of P_+^n . Then (2.9) and Hölder's inequality give

$$\begin{aligned} \int_K ||u(\vec{t})||_{\Sigma(\vec{A})} \omega(t) &\leq \int_{P_+^n} \min(1/\vec{t}) J(\vec{t}, u(\vec{t})) \omega(\vec{t}) \leq \\ &\leq \left(\int (|\vec{t}^{\vec{\theta}}| \min(1/\vec{t}))^{q'} \omega(t) \right)^{1/q'} \Phi_{\vec{\theta}q} [J(\vec{t}, u(\vec{t}))], \end{aligned}$$

where $1/q + 1/q' = 1$. Here, by (3.20'), the first factor is finite. Thus, if u obeys (4.8), then u is integrable with respect to ω .

Theorem 4.3. $\vec{A}_{\vec{\theta}q;J}$ is a Banach space under the norm

$$(4.9) \quad a \mapsto ||a||_{\vec{A}_{\vec{\theta}q;J}} = \inf_{\vec{\theta}q} \Phi_{\vec{\theta}q} [J(\vec{t}, u(\vec{t}); \vec{A})].$$

Here the infimum is taken over all functions u obeying (4.7) and (4.8). Furthermore

$$(4.10) \quad K(\vec{t}, a; \vec{A}) \leq C |\vec{t}^{\vec{\theta}}| ||a||_{\vec{A}_{\vec{\theta}q;J}} \quad \text{if } a \in \vec{A}_{\vec{\theta}q;J},$$

$$(4.11) \quad ||a||_{\vec{A}_{\vec{\theta}q;J}} \leq C |\vec{t}^{-\vec{\theta}}| J(\vec{t}, a; \vec{A}) \quad \text{if } a \in \Delta(\vec{A}).$$

Hence

$$(4.12) \quad \Delta(\vec{A}) \subset \vec{A}_{\vec{\theta}q;J} \subset \Sigma(\vec{A}).$$

Proof: The only non-trivial norm condition is : $||a|| = 0$ implies $a = 0$. This is however an immediate consequence of (4.10).

We verify (4.10). Let $a \in \vec{A}_{\vec{\theta}q;J}$, $u \in \mathcal{H}_0(\Delta(\vec{A}))$, with

$$a = \int u(\vec{s}) \omega(\vec{s}) \quad \text{in } \Sigma(\vec{A}).$$

Then

$$K(\vec{t}, a) \leq \int K(\vec{t}, u(\vec{s})) \omega(\vec{s}).$$

From (2.9) and Hölder's inequality we obtain

$$\begin{aligned} K(\vec{t}, a) &\leq \int \min(\vec{t}/\vec{s}) J(\vec{s}, u(\vec{s})) \omega(\vec{s}) \leq \\ &\leq \left(\int (|\vec{s}|^{\vec{\theta}} |\min(\vec{t}/\vec{s})|^{q'} \omega(\vec{s}))^{1/q'} \phi_{\vec{\theta}q} [J(\vec{s}, u(\vec{s}))] \right), \end{aligned}$$

where $1/q + 1/q' = 1$. Now (4.10) follows from (3.19) and (3.20').

To prove (4.11), let $\phi \in \mathcal{K}_0$, $\phi \geq 0$ and $\int \phi(\vec{t}) \omega(\vec{t}) = 1$.

With $a \in \Delta(\vec{A})$ we define

$$u(\vec{t}) = \phi(\vec{t}/\vec{s}) a.$$

Then, independently of $\vec{s}$, we have

$$a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}).$$

Thus, by (2.8) and (3.19),

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}q;J}} &\leq \phi_{\vec{\theta}q} [J(\vec{t}, u(\vec{t}))] = \\ &= \phi_{\vec{\theta}q} [\phi(\vec{t}/\vec{s}) J(\vec{t}, a)] \leq \phi_{\vec{\theta}q} [\phi(\vec{t}/\vec{s}) \max(\vec{t}/\vec{s})] J(\vec{s}, a) \\ &\leq \phi_{\vec{\theta}q} [\phi(\vec{t}) \max(\vec{t})] |\vec{s}|^{-\vec{\theta}} J(\vec{s}, a). \end{aligned}$$

Choosing ϕ so that the first factor is finite, we obtain (4.11).

Finally, the completeness follows exactly as in the case $n = 1$ (cf. the references given above). We omit the details. $\#$

Remark 4.4. If $q < \infty$, it is easily seen that $\Delta(\vec{A})$ is dense in $\vec{A}_{\vec{\theta}q;J}$. So need not necessarily be the case with

$\vec{A}_{\vec{\theta}q;K}$ (for a counter-example, see remark 5.1).

Also $\vec{A}_{\vec{\theta}q;J}$ is an interpolation space:

Theorem 4.4. The functor

$$(4.13) \quad \mathcal{L}_n \ni \vec{A} \mapsto \vec{A}_{\vec{\theta}q;J} \in \mathcal{L}_0$$

is an interpolation functor of type $|\vec{t}^{\vec{\theta}}|$.

Proof: Condition (i) in the definition of interpolation functors (def. 1.2) is already verified in view of (4.12). Next, consider a map $T : \vec{A} \rightarrow \vec{B}$. Let $\vec{M} = \vec{M}(T)$ be as in def.

1.3. Let $a \in \vec{A}_{\vec{\theta}_q; J}$, $u \in \mathcal{H}_0(\Delta(\vec{A}))$, with

$$a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}).$$

Then $Tu(\vec{t}) \in \mathcal{H}_0(\Delta(\vec{B}))$,

$$Ta = \int Tu(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{B}),$$

and

$$J(\vec{t}, Tu(\vec{t}); \vec{B}) \leq J(\vec{M}\vec{t}, u(\vec{t}); \vec{A}).$$

Now set $v(\vec{t}) = u(\vec{t}/\vec{M})$. Then $v \in \mathcal{H}_0(\Delta(\vec{A}))$,

$$a = \int v(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}),$$

and

$$J(\vec{t}, Tu(\vec{t}); \vec{B}) \leq J(\vec{M}\vec{t}, v(\vec{M}\vec{t}); \vec{A}).$$

Application of $\Phi_{\vec{\theta}_q}$ to this inequality, using also (3.19), then completes the proof. $\#$

We refer to the functor (4.13) as the J-functor (or J-method).

Remark 4.5. It is also possible to give a discrete characterization of $\vec{A}_{\vec{\theta}_q; K}$ and $\vec{A}_{\vec{\theta}_q; J}$, using sums instead of integrals in (4.2) and (4.7), (4.8). Let $\vec{v} = (v_0, v_1, \dots, v_n) \in \mathbb{Z}^{n+1}$. Because of the monotonicity of the functions involved, we get for example that

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}_q; K}} &= \left(\int_{\min(\vec{t})=1} (|\vec{t}^{-\vec{\theta}}| K(\vec{t}, a; \vec{A}))^q \omega(\vec{t}) \right)^{1/q} \\ &\approx \left(\sum_{\min(\vec{v})=0} (|2^{-\vec{v}\vec{\theta}}| K(2^{\vec{v}}, a; \vec{A}))^q \right)^{1/q}. \end{aligned}$$

In the J-case, we rewrite (4.7) as

$$a = \int_{\min(\vec{t})=1} u(\vec{t}) \omega(\vec{t}) = \sum_{\min(\vec{v})=0} u_{\vec{v}}$$

with

$$u_{\vec{v}} = \int_{\substack{\min(\vec{t})=1 \\ 2^{\vec{v}} \leq \vec{t} \leq 2^{\vec{v}+1}}} u(\vec{t}) \omega(\vec{t}).$$

Then $u_{\vec{v}} \in \Delta(\vec{A})$. On the other hand, every such sequence defines in a natural way a function in $\mathcal{H}_0(\Delta(\vec{A}))$, obeying (4.7). Thus

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}q;J}} &= \inf \left(\int (|\vec{t}^{-\vec{\theta}}| J(\vec{t}, u(\vec{t}); \vec{A}))^q \omega(\vec{t}) \right)^{1/q} = \\ &= \inf \left(\sum_{\min(\vec{v})=0} (|2^{-\vec{v}\vec{\theta}}| J(2^{\vec{v}}, u_{\vec{v}}; \vec{A}))^q \right)^{1/q}. \end{aligned}$$

These facts will be used below, in the proof of th. 9.2.

For a future reference (lemma 10.1) we write down another equivalent norm of $\vec{A}_{\vec{\theta}q;K}$. This time we consider the P-surface

$t_0 = 1$. Let $\vec{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ with $0 < \gamma_i \leq 1$ ($i = 1, 2, \dots, n$). We write $(1, \vec{\gamma}^{\vec{\mu}}) = (1, \gamma_1^{\mu_1}, \dots, \gamma_n^{\mu_n})$, where $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$.

Then

$$\|a\|_{\vec{A}_{\vec{\theta}q;K}} \approx \left(\sum_{\vec{\mu} \in \mathbb{Z}^n} (\gamma_1^{-\mu_1 \theta_1} \gamma_2^{-\mu_2 \theta_2} \dots \gamma_n^{-\mu_n \theta_n} K((1, \vec{\gamma}^{\vec{\mu}}), a; \vec{A}))^q \right)^{1/q}.$$

Remark 4.6. Starting from Lions-Peetre [13], Yoshikawa [25] defines the spaces $S(\vec{q}; \vec{\xi}; \vec{A})$ and $\underline{S}(\vec{q}; \vec{\xi}; \vec{A})$. In a form easy to compare with ours, these definitions can be restated in the following way:

i) $a \in \vec{A}_{\vec{\theta}q;K}$ if and only if

$$\|a\|_{\vec{A}_{\vec{\theta}q;K}} = \inf \sum_{i=0}^n \left(\int (|\vec{t}^{-\vec{\theta}}| t_i \|a_i(\vec{t})\|_{A_i})^{q_i} \omega(\vec{t}) \right)^{1/q_i} < \infty.$$

Here the infimum is taken over all $a_i \in \mathcal{H}_0(A_i)$ ($i = 0, 1, \dots, n$) such that $a = a_0(\vec{t}) + a_1(\vec{t}) + \dots + a_n(\vec{t})$ a.e.

ii) $a \in \vec{A}_{\vec{\theta}q;J}$ if and only if

$$\|a\|_{\vec{A}_{\vec{\theta}q;J}} = \inf \max_{i=0,1,\dots,n} (\int (\|\vec{t}^{-\vec{\theta}}\|_{A_i} \|u(\vec{t})\|_{A_i})^{q_i} \omega(\vec{t}))^{1/q_i} < \infty.$$

Here the infimum is taken over all $u \in \mathcal{H}_0(\Delta(\vec{A}))$ such that $a = \int u(\vec{t}) \omega(\vec{t})$ in $\Sigma(\vec{A})$.

In the case $n = 1$, it was proved by Peetre (see Peetre [17]) that here the number of independent parameters can be reduced. In fact, if $1/q = \theta_0/q_0 + \theta_1/q_1$, then

$$(4.15) \quad \vec{A}_{\vec{\theta}q;K} = \vec{A}_{\vec{\theta}q;K},$$

$$(4.16) \quad \vec{A}_{\vec{\theta}q;J} = \vec{A}_{\vec{\theta}q;J}$$

(the parameter theorem). For $n > 1$, by a generalization of this proof, (4.16) still holds true, with $1/q = \sum_{i=0}^n \theta_i/q_i$. So is also the case with (4.15). This can be proved by an argument, analogous to that of Peetre [22], also using the results of Section 7 below. (For the K-case, cf. also Holmstedt [8]).

Remark 4.7. There is also an obvious possibility to define n -functors by "composition" of 1-functors. By this we mean functors corresponding to spaces like

$$((A_0, A_1)_{\vec{\theta}_1 q_1}, A_2)_{\vec{\theta}_2 q_2},$$

$$((A_0, A_1)_{\vec{\theta}_1 q_1}, (A_0, A_2)_{\vec{\theta}_2 q_2})_{\vec{\theta}_3 q_3},$$

$$((A_0, (A_1, A_2)_{\vec{\theta}_1 q_1})_{\vec{\theta}_2 q_2}, A_3)_{\vec{\theta}_3 q_3} \quad \text{etc.}$$

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(In the case $n = 1$, as we mentioned above, one need not distinguish between K- and J-spaces). It is easily seen that every such functor is an interpolation functor of type $|\vec{t}^{\vec{\theta}}|$ for some $\vec{\theta}$.

Let $\vec{A} \in \mathcal{C}_n$, $n > 1$. Consider the interpolation spaces with respect to $\vec{A}$, formed in this way. There is a multitude of such spaces. In an important sense they still are contained in our treatment of K- and J-spaces (remark 9.4). On the other hand, on "wellbehaved" tuples every K- and J-functor can be expressed in this way, by use of 1-functors only (th. 9.3).

This alternative approach to interpolation of several spaces however fails in important respects. Indeed, even in very simple cases one meets great difficulties (remark 10.3). Trying to derive more general results this way, for example analogues of the stability results of Section 9, there seems to be little hope for success.

[illegible]

John W. ...

5. On the condition $\mathcal{F}$ and the equivalence of K- and J-spaces.

We recall that if $n = 1$ the spaces $\vec{A}_{\vec{\theta}q;K}$ and $\vec{A}_{\vec{\theta}q;J}$ coincide algebraically and topologically (the equivalence theorem, see Peetre [16] or Butzer-Berens [5]). For $n > 1$ the situation is not so pleasant. Then in general only the inclusion $\vec{A}_{\vec{\theta}q;J} \subset \vec{A}_{\vec{\theta}q;K}$ is valid (prop. 5.1), but not the converse (remark 5.1). This defect is caused by the fact, that for $n > 1$ we have no analogue of the fundamental lemma: If

$$(5.1) \quad \int \min(1, 1/t) K((1, t), a) \frac{dt}{t} < \infty,$$

then there exists a function $u = u(t)$ with values in $\Delta(\vec{A})$ such that

$$(5.2) \quad a = \int u(t) \frac{dt}{t} \quad \text{in } \Sigma(\vec{A}),$$

$$(5.3) \quad J((1, t), u(t)) \leq \gamma K((1, t), a).$$

Here γ is independent of a and t . For a proof, we refer to Peetre [16], Ch. V, lemma 1 or Butzer-Berens [5], lemma 3.2.10.

However, in many situations one can show that the analogue of the fundamental lemma really holds true. We describe this by saying that "the condition $\mathcal{F}$ is satisfied" (def. 5.2). In that case the K- and J-spaces are equal (th. 5.1).

Concerning the fundamental lemma, the Appendix serves to illuminate what happens if $n > 1$.

Let us now carry out this program, starting with

Proposition 5.1. Let $1 \leq q \leq \infty$, $\vec{\theta} \in H_+^n$. Then

$$\vec{A}_{\vec{\theta}q;J} \subset \vec{A}_{\vec{\theta}q;K}$$

Proof: Let $a \in \vec{A}_{\vec{\theta}q;J}$, $u \in \mathcal{D}_0(\Delta(\vec{A}))$ with

$$a = \int u(\vec{s}) \omega(\vec{s}) \quad \text{in } \Sigma(\vec{A}).$$

Then, by (2.9),

$$\begin{aligned} K(\vec{t}, a) &\leq \int K(\vec{t}, u(\vec{s})) \omega(\vec{s}) \leq \int \min(\vec{t}/\vec{s}) J(\vec{s}, u(\vec{s})) \omega(\vec{s}) = \\ &= \int \min(1/\vec{s}) J(\vec{s}\vec{t}, u(\vec{s}\vec{t})) \omega(\vec{s}). \end{aligned}$$

Application of $\Phi_{\vec{\theta}q}$ yields, by means of Minkowski's inequality,

$$\Phi_{\vec{\theta}q} [K(\vec{t}, a)] \leq (\int |\vec{s}^{\vec{\theta}}| \min(1/\vec{s}) \omega(\vec{s})) \Phi_{\vec{\theta}q} [J(t, u(t))].$$

By (3.20') the first factor is finite. Making u vary, we obtain the desired inequality

$$\|a\|_{\vec{A}_{\vec{\theta}q;K}} \leq C \|a\|_{\vec{A}_{\vec{\theta}q;J}}. \quad \#$$

Remark 5.1. The following example shows that the converse inclusion $\vec{A}_{\vec{\theta}q;K} \subset \vec{A}_{\vec{\theta}q;J}$ does not hold true in general (cf. also Yoshikawa [25]).

Let A_0 and A_1 be two Banach spaces such that

$$A_1 \subset (A_0, A_1)_{\vec{\mu}^\infty; K} \subset A_0, \quad \vec{\mu} = (\mu_0, \mu_1) \in H_+^1$$

where the first inclusion is supposed to be proper. Choose

$$a \in (A_0, A_1)_{\vec{\mu}^\infty; K} \text{ such that } a \notin A_1.$$

Let A_2 be the (one dimensional) vector space generated by a .

The norm on A_2 is $\|b\|_{A_2} = \|b\|_{A_0}$, $b \in A_2$. Now consider

the Banach triple $\vec{A} = \{A_0, A_1, A_2\}$. Since $\Delta(\vec{A}) = 0$, we have

$\vec{A}_{\vec{\theta}q;J} = 0$ for every $\vec{\theta}$ (and q). However, we shall see

that $\vec{A}_{\vec{\theta}q;K} \neq 0$ for suitable values of $\vec{\theta}$.

In fact, we shall prove that $a \in \vec{A}_{\vec{\theta}q;K}$ for some $\vec{\theta}$. With-

out loss of generality, we can assume that $\|a\|_{A_0} = \|a\|_{A_2} = 1$,

$$\|a\|_{(A_0, A_1)_{\vec{\mu}^\infty; K}} \leq 1. \text{ Then}$$

$$\begin{aligned} K(\vec{t}, a; \vec{A}) &\leq \min(t_0 \|a\|_{A_0}, K((t_0, t_1)a; \{A_0, A_1\}), t_2 \|a\|_{A_2}) \\ &\leq \min(t_0, K((t_0, t_1), a; \{A_0, A_1\}), t_2). \end{aligned}$$

Here, by assumption,

$$K((t_0, t_1), a; \{A_0, A_1\}) \leq t_0^{\mu_0} t_1^{\mu_1}.$$

Hence

$$K(\vec{t}, a; \vec{A}) \leq \min(t_0, t_0^{\mu_0} t_1^{\mu_1}, t_2).$$

We will now use prop. 3.2. Consider the automorphism on P_+^2

$$\alpha(\vec{t}) = (t_0, t_0^{\mu_0} t_1^{\mu_1}, t_2).$$

To get agreement with the assumptions of prop. 3.2 we put

$$\begin{aligned} \vec{\theta}^0 &= (1, 0, 0), \\ \vec{\theta}^1 &= (\mu_0, \mu_1, 0), \\ \vec{\theta}^2 &= (0, 0, 1). \end{aligned}$$

Furthermore, let

$$\begin{aligned} \vec{\theta} &= (\theta_0, \theta_1, \theta_2) \in H_+^2, \\ \vec{\lambda} &= (\lambda_0, \lambda_1, \lambda_2) = (1 - \theta_1/\mu_1 - \theta_2, \theta_1/\mu_1, \theta_2). \end{aligned}$$

Then

$$\vec{\theta} = \lambda_0 \vec{\theta}^0 + \lambda_1 \vec{\theta}^1 + \lambda_2 \vec{\theta}^2.$$

Assume that $\theta_1/\mu_1 + \theta_2 \leq 1$. Then $\vec{\lambda} \in \bar{H}_+^n$. From (3.24) of prop. 3.2 we obtain, using also (3.20),

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}_q; K}} &= \Phi_{\vec{\theta}_q} [K(\vec{t}, a; \vec{A})] \leq \Phi_{\vec{\theta}_q} [\min(t_0, t_0^{\mu_0} t_1^{\mu_1}, t_2)] \\ &= C \Phi_{\vec{\lambda}_q} [\min(\vec{t})] < \infty. \end{aligned}$$

We have thus proved that $a \in \vec{A}_{\vec{\theta}_q; K}$ if $\theta_1/\mu_1 + \theta_2 \leq 1$. Hence

$\vec{A}_{\vec{\theta}_q; K} \neq \vec{A}_{\vec{\theta}_q; J}$ in this case. $\#$

In the next two definitions we introduce a sufficient condition for equality between the K- and J-spaces (cf. the fundamental lemma referred to above).

Definition 5.1. By $\sigma(\vec{A})$ we mean the set of all $a \in \Sigma(\vec{A})$ for which

$$(5.4) \quad \int \min(1/\vec{t}) K(\vec{t}, a; \vec{A}) \omega(t) < \infty.$$

Remark 5.2. If $1 \leq q \leq \infty$, $\vec{\theta} \in H_+^n$, then $\vec{A}_{\vec{\theta}q;K}$ and $\vec{A}_{\vec{\theta}q;J}$ are contained in $\sigma(\vec{A})$. This follows from (4.3) and (4.10), in view of (3.20').

Definition 5.2. We say that $\vec{A}$ satisfies the condition $\widetilde{\mathcal{F}}$, or that $\widetilde{\mathcal{F}}(\vec{A})$ is satisfied, if for every $a \in \sigma(\vec{A})$ there exists a function $u \in \mathcal{H}_0(\Delta(\vec{A}))$, such that

$$(5.5) \quad a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}),$$

$$(5.6) \quad J(\vec{t}, u(\vec{t}); \vec{A}) \leq \gamma(\vec{A}) K(\vec{t}, a; \vec{A}).$$

Here $\gamma(\vec{A})$ depends only on $\vec{A}$, not on a or $\vec{t}$.

Remark 5.3. Actually, we need only require that (5.5) holds with convergence in some weaker sense, for example as a generalized integral. The absolute convergence is namely a consequence of (5.6). This is seen by an argument, similar to that of remark 4.3.

In the case $n = 1$, $\widetilde{\mathcal{F}}(\vec{A})$ is satisfied for any Banach couple $\vec{A}$ (the fundamental lemma). If $n > 1$ this is not the case in general. For example, the triple $\vec{A}$ of remark 5.1 does not satisfy the condition $\widetilde{\mathcal{F}}$. This is a consequence of the following theorem:

Theorem 5.1. (The equivalence theorem). Let $1 \leq q \leq \infty$, $\vec{\theta} \in H_+^n$. If $\widetilde{\mathcal{F}}(\vec{A})$ is satisfied, then

$$\vec{A}_{\vec{\theta}q;K} = \vec{A}_{\vec{\theta}q;J}$$

Proof: In view of prop. 5.1, only the inclusion $\vec{A}_{\vec{\theta}_q;K} \subset \vec{A}_{\vec{\theta}_q;J}$ remains to be verified. To this end, let $a \in \vec{A}_{\vec{\theta}_q;K}$. Then $a \in \sigma(\vec{A})$ (remark 5.2). Application of $\Phi_{\vec{\theta}_q}$ to (5.6) yields

$$\|a\|_{\vec{A}_{\vec{\theta}_q;J}} \leq \Phi_{\vec{\theta}_q} [J(\vec{t}, u(\vec{t}))] \leq \gamma \Phi_{\vec{\theta}_q} [K(t, a)] = \gamma \|a\|_{\vec{A}_{\vec{\theta}_q;K}}.$$

This is the desired inequality. $\#$

In this theorem, and throughout the paper, by saying that two spaces are equal we mean that they coincide algebraically and have equivalent norms. If, for a particular $\vec{A}$, the K - and J -spaces are equal, we agree to drop the letters K and J , writing

$$\vec{A}_{\vec{\theta}_q;K} = \vec{A}_{\vec{\theta}_q;J} = \vec{A}_{\vec{\theta}_q}.$$

Conversely, if we write $\vec{A}_{\vec{\theta}_q}$, it is to be understood that $\vec{A}_{\vec{\theta}_q;K} = \vec{A}_{\vec{\theta}_q;J}$.

In the sequel, we will consider several concrete situations where condition $\mathcal{F}$ can be verified. For example, this can be done for certain tuples of L_p -spaces with weights (prop. 8.1). This fact will be of importance in the applications of Section 10. There we give other examples where $\mathcal{F}$ is satisfied.

Throughout the paper, we will pay some attention to what can be said about condition $\mathcal{F}$ at the different stages. We shall give several examples of functors under which $\mathcal{F}$ is preserved, see e.g. prop. 6.4, 8.1, 8.2 and 9.3. These facts will then be used in Section 10.

Another such result is given by the following proposition. In the case $n = 1$ it can be found in Peetre [21]. We recall the following definition (cf. Peetre [23]): Let $\vec{A}, \vec{B} \in \mathcal{C}_n$. $\vec{B}$ is said to be a retract of $\vec{A}$ if there exist morphisms P

(1) In the case of a function f of n variables, f is said to be k -linear if it is linear in each of its k arguments. For example, a function of two variables is k -linear if it is linear in each of its two arguments.

(2) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(3) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(4) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(5) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(6) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(7) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(8) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(9) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

(10) Let f be a function of n variables. We say that f is k -linearly separable if there exist k functions $f_1, \dots, f_k$ of n variables such that f is the sum of these functions. For example, a function of two variables is k -linearly separable if it can be written as the sum of k functions of two variables.

and I such that the diagram

$$(5.7) \quad \begin{array}{ccc} \vec{B} & \xrightarrow{\text{id}} & \vec{B} \\ I \searrow & & \nearrow P \\ & \vec{A} & \end{array}$$

is commutative (i.e. $P \circ I = \text{id}$).

Proposition 5.2. Let $\vec{B}$ be a retract of $\vec{A}$. If $\mathcal{F}(\vec{A})$ is satisfied, then also $\mathcal{F}(\vec{B})$ is satisfied.

Proof: Let P and I be given by (5.7). Then

$$(5.8) \quad K(\vec{t}, Ib; \vec{A}) \leq C K(\vec{t}, b; \vec{B}),$$

$$(5.9) \quad J(\vec{t}, b; \vec{B}) \leq C J(\vec{t}, Ib; \vec{A}).$$

Let $b \in \sigma(\vec{B})$. Then $Ib \in \sigma(\vec{A})$, in view of (5.8). By assumption there exists a function $u \in \mathcal{B}_0(\Delta(\vec{A}))$ such that

$$Ib = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}),$$

$$J(\vec{t}, u(\vec{t}); \vec{A}) \leq C K(\vec{t}, Ib; \vec{A}).$$

But then

$$b = PIb = \int Pu(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{B}).$$

Moreover, by (5.8) and (5.9),

$$\begin{aligned} J(\vec{t}, Pu(\vec{t}); \vec{B}) &\leq C J(\vec{t}, u(\vec{t}); \vec{A}) \leq \\ &\leq C K(\vec{t}, Ib; \vec{A}) \leq C K(\vec{t}, b; \vec{B}). \end{aligned}$$

Hence $\mathcal{F}(\vec{B})$ is satisfied. $\#$

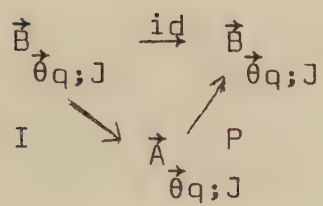
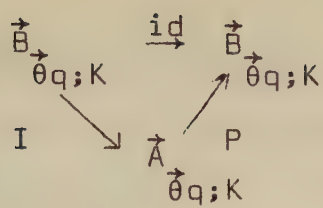
By interpolation in (5.7) we immediately get

Proposition 5.3. Let $\vec{B}$ be a retract of $\vec{A}$. Then $\vec{B}_{\vec{\theta}_q; K}$ is

a retract of $\vec{A}_{\vec{\theta}_q; J}$. Moreover, if

$$\begin{array}{ccc} \vec{B} & \xrightarrow{\text{id}} & \vec{B} \\ I \searrow & & \nearrow P \\ & \vec{A} & \end{array}$$

then, with the same P and I ,



#

6. Various properties of K- and J-spaces.

We begin with (cf. Butzer-Berens [5] for the case $n = 1$):

Proposition 6.1. If $1 \leq q_1 \leq q_2 \leq \infty$, then

$$(6.1) \quad \vec{A}_{\vec{\theta}_{q_1}; K} \subset \vec{A}_{\vec{\theta}_{q_2}; K},$$

$$(6.2) \quad \vec{A}_{\vec{\theta}_{q_1}; J} \subset \vec{A}_{\vec{\theta}_{q_2}; J}$$

Proof: We will use the obvious inequality

$$(6.3) \quad \Phi_{\vec{\theta}_{q_2}}[f] \leq (\Phi_{\vec{\theta}_{\infty}}[f])^{1-q_1/q_2} (\Phi_{\vec{\theta}_{q_1}}[f])^{q_1/q_2}.$$

We apply this inequality to $K(\vec{t}, a; \vec{A})$. In view of (4.3), we have

$$\Phi_{\vec{\theta}_{\infty}}[K(\vec{t}, a)] \leq C \Phi_{\vec{\theta}_{q_1}}[K(\vec{t}, a)].$$

Taking this into account, the first inclusion is verified. To

verify (6.2), let $a \in \vec{A}_{\vec{\theta}_{q_1}; J}$, $u \in \mathcal{H}_0(\Delta(\vec{A}))$, with

$$a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}).$$

Let ϕ be a positive function on P_+^n such that

$$\int \phi(\vec{t}) \omega(\vec{t}) = 1.$$

Put

$$v(\vec{t}) = \int \phi(\vec{t}/\vec{s}) u(\vec{s}) \omega(\vec{s}).$$

Then $v \in \mathcal{H}_0(\Delta(\vec{A}))$ and

$$a = \int v(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}).$$

Furthermore, by (2.7) and Hölder's inequality,

$$\begin{aligned} J(\vec{t}, v(\vec{t})) &\leq \int \phi(\vec{t}/\vec{s}) J(\vec{t}, u(\vec{s})) \omega(\vec{s}) \leq \\ &\leq \int \phi(\vec{t}/\vec{s}) \max(\vec{t}/\vec{s}) J(\vec{s}, u(\vec{s})) \omega(\vec{s}) \leq \\ &\leq \left(\int (|\vec{s}|^{\vec{\theta}} |\phi(\vec{t}/\vec{s}) \max(\vec{t}/\vec{s})|^{q'_1} \omega(\vec{s}))^{1/q'_1} \Phi_{\vec{\theta}_{q_1}}[J(\vec{s}, u(\vec{s}))] \right), \end{aligned}$$

where $1/q + 1/q' = 1$.

Let ϕ be so chosen that

$$(\int (|\vec{s}^{-\vec{\theta}}| \phi(\vec{s}) \max(\vec{s}))^{q'_1 \omega(\vec{s})})^{1/q'_1} < \infty.$$

Then

$$J(\vec{t}, v(\vec{t})) \leq C \cdot |\vec{t}^{-\vec{\theta}}|_{\phi_{\vec{\theta}_{q_1}}} [J(\vec{s}, u(\vec{s}))],$$

or equivalently,

$$\phi_{\vec{\theta}_{\infty}} [J(\vec{t}, v(\vec{t}))] \leq C \phi_{\vec{\theta}_{q_1}} [J(\vec{s}, u(\vec{s}))].$$

By application of (6.3) to the function $J(\vec{t}, v(\vec{t}))$ and variation of v (and u), (6.2) is verified. $\#$

Concerning the reversal of the order of the spaces we have the obvious

Proposition 6.2. Let π be a permutation of $\{0, 1, \dots, n\}$ and let $\vec{A}^{\pi} = \{A_{\pi(0)}, A_{\pi(1)}, \dots, A_{\pi(n)}\}$, $\vec{\theta}^{\pi} = (\theta_{\pi(0)}, \theta_{\pi(1)}, \dots, \theta_{\pi(n)})$. Then

$$\vec{A}_{\vec{\theta}_{q;K}} = \vec{A}^{\pi}_{\vec{\theta}^{\pi}_{q;K}},$$

$$\vec{A}_{\vec{\theta}_{q;J}} = \vec{A}^{\pi}_{\vec{\theta}^{\pi}_{q;J}}. \quad \#$$

If two (or more) of the spaces are equal, it is possible to reduce to interpolation of lower order. This can be done by repeated use of prop. 6.2 combined with the following:

Proposition 6.3. If $A_{n-1} = A_n$, then

$$\vec{A}_{\vec{\theta}_{q;K}} = (A_0, A_1, \dots, A_{n-1}) (\theta_0, \theta_1, \dots, (\theta_{n-1} + \theta_n))_{q;K},$$

$$\vec{A}_{\vec{\theta}_{q;J}} = (A_0, A_1, \dots, A_{n-1}) (\theta_0, \theta_1, \dots, (\theta_{n-1} + \theta_n))_{q;J}.$$

Proof: We restrict ourselves to the K-case. Let

$\vec{A}' = \{A_0, A_1, \dots, A_{n-1}\}$. Omitting for a moment the variables $t_0, t_1, \dots, t_{n-2}$, we set

$$k(t_{n-1}, t_n) = K(\vec{t}, a; \vec{A}),$$

$$k'(t_{n-1}) = K((t_0, t_1, \dots, t_{n-1}), a; \vec{A}').$$

Then

$$k(t_{n-1}, t_n) = k'(\min(t_{n-1}, t_n)).$$

A simple calculation yields

$$\int (t_{n-1}^{-\theta_{n-1}} t_n^{-\theta_n} k(t_{n-1}, t_n))^q \frac{dt_{n-1}}{t_{n-1}} \frac{dt_n}{t_n} = \\ = C \int (s^{-(\theta_{n-1} + \theta_n)} k'(s))^q \frac{ds}{s}.$$

Let $t_0 = 1$. By an integration in the variables $t_1, t_2, \dots, t_{n-2}$ we obtain

$$\int_{t_0=1} (|\vec{t}|^{-\vec{\theta}} |K(\vec{t}, a; \vec{A})|)^q \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n} = \\ = C \int (t_1^{-\theta_1}, \dots, t_{n-1}^{-(\theta_{n-1} + \theta_n)} K((1, t_1, \dots, t_{n-1}), a; \vec{A}'))^q \\ \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_{n-1}}{t_{n-1}}.$$

In view of prop. 3.1, this concludes the proof. $\#$

Proposition 6.4. Let $A_{n-1} = A_n$. If $\mathcal{F}(\{A_0, A_1, \dots, A_{n-1}\})$ is satisfied, then also $\mathcal{F}(\vec{A})$ is satisfied.

Proof: Let $\vec{A}' = \{A_0, A_1, \dots, A_{n-1}\}$. We have the relations

$$(6.4) \quad K(\vec{t}, a; \vec{A}) = K((t_0, t_1, \dots, t_{n-2}, \min(t_{n-1}, t_n)), a; \vec{A}'),$$

$$(6.5) \quad J(\vec{t}, a; \vec{A}) = K((t_0, t_1, \dots, t_{n-2}, \max(t_{n-1}, t_n)), a; \vec{A}').$$

Let $s_i = t_i$ ($i = 0, 1, \dots, n-2$), $s_{n-1} = \max(t_{n-1}, t_n)$. We will integrate over the P-surface (in R_+^{n+1}) $\max(t_{n-1}, t_n) = 1$. In R_+^n this corresponds to the P-surface $s_{n-1} = 1$. On $\max(t_{n-1}, t_n) = 1$ holds

$$\omega(\vec{t}) = (-1)^{n-1} \frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_{n-2}}{t_{n-2}} \wedge \left(\frac{dt_n}{t_n} - \frac{dt_{n-1}}{t_{n-1}} \right).$$

But here

$$\frac{dt_0}{t_0} \wedge \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_{n-2}}{t_{n-2}} = \frac{ds_0}{s_0} \wedge \frac{ds_1}{s_1} \wedge \dots \wedge \frac{ds_{n-2}}{s_{n-2}},$$

which is the expression for $\omega(\vec{s})$ on $s_{n-1} = 1$. First we prove that $\sigma(\vec{A}) \subset \sigma(\vec{A}')$. Let $a \in \sigma(\vec{A})$. From (6.4) we get

$$K(\vec{s}, a; \vec{A}') \leq \frac{\max(t_{n-1}, t_n)}{\min(t_{n-1}, t_n)} K(\vec{t}, a; \vec{A}).$$

We note that

$$\int_{\max(t_{n-1}, t_n)=1} \min(t_{n-1}, t_n) \left(\frac{dt_n}{t_n} - \frac{dt_{n-1}}{t_{n-1}} \right) = 2 ,$$

$$\max(\vec{s}) = \max(\vec{t}) .$$

Hence

$$\int_{s_{n-1}=1} \frac{K(\vec{s}, a; \vec{A}')}{\max(\vec{s})} \omega(\vec{s}) = \frac{1}{2} \left\{ \int_{s_{n-1}=1} \frac{K(\vec{s}, a; \vec{A}')}{\max(\vec{s})} \omega(\vec{s}) \right\} \cdot$$

$$\cdot \left\{ \int_{\max(t_{n-1}, t_n)=1} \min(t_{n-1}, t_n) \left(\frac{dt_n}{t_n} - \frac{dt_{n-1}}{t_{n-1}} \right) \right\} \leq$$

$$\leq \frac{1}{2} \int_{\max(t_{n-1}, t_n)=1} \frac{K(\vec{t}, a; \vec{A})}{\max(\vec{t})} \omega(\vec{t}) .$$

Thus $a \in \sigma(\vec{A}')$. Then by assumption there exists a function

$u = u(\vec{s}) \in \mathcal{H}_0(\Delta(\vec{A}'))$ such that

$$a = \int u(\vec{s}) \omega(\vec{s}) \quad \text{in} \quad \Sigma(\vec{A}'),$$

$$J(\vec{s}, u(\vec{s}), \vec{A}') \leq \gamma K(\vec{s}, a; \vec{A}').$$

Let the function $v = v(\vec{t})$ in $\mathcal{H}_0(\Delta(\vec{A}))$ be defined by

$$2 \cdot \frac{\max(t_{n-1}, t_n)}{\min(t_{n-1}, t_n)} v(\vec{t}) = u(\vec{s}).$$

Then, arguing as above,

$$\int_{\max(t_{n-1}, t_n)=1} v(\vec{t}) \omega(\vec{t}) =$$

$$= \int_{s_{n-1}=1} u(\vec{s}) \omega(\vec{s}) \int_{\max(t_{n-1}, t_n)=1} \frac{1}{2} \min(t_{n-1}, t_n) \left(\frac{dt_n}{t_n} - \frac{dt_{n-1}}{t_{n-1}} \right) =$$

$$= \int u(\vec{s}) \omega(\vec{s}) = a \quad \text{in} \quad \Sigma(\vec{A}).$$

Finally, by (6.4) and (6.5),

$$J(\vec{t}, v(\vec{t}), \vec{A}) \leq \frac{1}{2} \frac{\min(t_{n-1}, t_n)}{\max(t_{n-1}, t_n)} J(\vec{s}, u(\vec{s}), \vec{A}') \leq$$

$$\leq \gamma \frac{\min(t_{n-1}, t_n)}{\max(t_{n-1}, t_n)} K(\vec{s}, a; \vec{A}') \leq \gamma K(\vec{t}, a; \vec{A}).$$

This completes the proof. $\#$

7. The power theorem.

In this section we essentially follow Peetre [22], where the case $n = 1$ is treated. However, our proofs are simpler even when confined to this special-case.

Let $A^{[q]}$, $0 < q < \infty$, denote the Banach space A with its norm $|| \cdot ||_A$ replaced by the functional

$$(7.1) \quad a \mapsto ||a||_A^q.$$

This functional does in general not define a (vector space) norm on A , but will still be useful to us later (in connection with L_q -spaces, Section 8). Within the framework of an interpolation theory for normed Abelian groups (which goes entirely parallel to the Banach space theory) such functionals can be used more systematically (cf. Peetre-Sparr [24] for the case $n = 1$).

If $\vec{A} = \{A_0, A_1, \dots, A_n\}$ is a Banach tuple, $\vec{q} = (q_0, q_1, \dots, q_n)$, $0 < q_i < \infty$ ($i = 0, 1, \dots, n$) we set

$$\vec{A}^{[\vec{q}]} = \{A_0^{q_0}, A_1^{q_1}, \dots, A_n^{q_n}\}.$$

If $q_0 = q_1 = \dots = q_n = 1$, as before, we write $\vec{A}$ instead of $\vec{A}^{[\vec{q}]}$.

Definition 7.1. We define

$$(7.2) \quad K(\vec{t}, a; \vec{A}^{[\vec{q}]}) = \inf(t_0 ||a_0||_{A_0}^{q_0} + t_1 ||a_1||_{A_1}^{q_1} + \dots + t_n ||a_n||_{A_n}^{q_n}).$$

Here the infimum is taken over all decompositions $a = a_0 + a_1 + \dots + a_n$, $a_i \in A_i$ ($i = 0, 1, \dots, n$).

Using $K(\vec{t}, a; \vec{A}^{[\vec{q}]})$ instead of $K(\vec{t}, a; \vec{A})$ in def. 4.1 we get:

Definition 7.2. By $(\vec{A}^{[\vec{q}]})_{\vec{\theta}_q; K}$ we denote the set of all

$a \in \Sigma(\vec{A})$ for which

$$||a||_{(\vec{A}^{[\vec{q}]})_{\vec{\theta}_q; K}} = \Phi_{\vec{\theta}_q} [K(\vec{t}, a; \vec{A}^{[\vec{q}]})] < \infty.$$

(Note that $\| \cdot \|_{(\vec{A}[\vec{q}])_{\vec{\theta}_{q;K}}}$ does not in general define a norm).

Instead of the K - functional we could have used an equivalent functional, thus obtaining an equivalent "norm". This is proved in the following proposition, which will be important in the proof of th. 7.1 below.

Proposition 7.1. Given $C > 1$, there exists a functional $\kappa(\vec{t}, a; \vec{A}[\vec{q}])$ on $\Sigma(\vec{A})$, differentiable with respect to $\vec{t}$, such that for any $a \in \Sigma(\vec{A})$

$$(7.3) \quad \frac{1}{C} \kappa(\vec{t}, a; \vec{A}[\vec{q}]) \leq \kappa(\vec{t}, a; \vec{A}[\vec{q}]) \leq C \kappa(\vec{t}, a; \vec{A}[\vec{q}]) .$$

Proof: Let $\phi \in \mathcal{H}_1$ be a non-negative differentiable function on R_+^{n+1} , vanishing outside some cone with vertex in the origin. We define

$$\kappa(\vec{t}, a; \vec{A}[\vec{q}]) = \int \phi(\vec{t}/\vec{s}) \kappa(\vec{s}, a; \vec{A}[\vec{q}]) \omega(\vec{s}).$$

(Note that the integrand is homogeneous in the $\vec{s}$ -variable since $\phi \in \mathcal{H}_1$). Then κ is differentiable and belongs to $\mathcal{H}_1$. By an analogue of (2.7) we obtain

$$\begin{aligned} \kappa(\vec{t}) \int \min(\vec{s}/\vec{t}) \phi(\vec{t}/\vec{s}) \omega(\vec{s}) &\leq \kappa(\vec{t}) \leq \\ &\leq \kappa(\vec{t}) \int \max(\vec{s}/\vec{t}) \phi(\vec{t}/\vec{s}) \omega(\vec{s}). \end{aligned}$$

Now (7.3) will follow if we choose ϕ to obey

$$1/C \leq \int \min(1/\vec{s}) \phi(\vec{s}) \omega(\vec{s}) \leq \int \max(1/\vec{s}) \phi(\vec{s}) \omega(\vec{s}) \leq C. \quad \#$$

It is always possible to reduce to the case $q_0 = q_1 = \dots = q_n = 1$:

Proposition 7.2. If either

$$(7.4) \quad s_i = t_i^{q_i} (\kappa(\vec{t}, a; \vec{A}))^{1-q_i} \quad (i = 0, 1, \dots, n)$$

or

$$(7.5) \quad t_i = s_i^{1/q_i} (\kappa(\vec{s}, a; \vec{A}[\vec{q}]))^{1-1/q_i} \quad (i = 0, 1, \dots, n),$$

then

$$(7.6) \quad C_1 \kappa(\vec{t}, a; \vec{A}) \leq \kappa(\vec{s}, a; \vec{A}[\vec{q}]) \leq C_2 \kappa(\vec{t}, a; \vec{A}).$$

Here C_1 and C_2 are independent of a and $\vec{t}$.

Proof: We introduce the functional

$$K_x(\vec{t}, a; \vec{A}[\vec{q}]) = \inf \max_{i=0,1,\dots,n} t_i \|a_i\|_{A_i}^{q_i}.$$

Obviously $K_x(\vec{t}, a; \vec{A}[\vec{q}])$ is equivalent to $K(\vec{t}, a; \vec{A}[\vec{q}])$. In particular we have

$$K_x(\vec{t}, a; \vec{A}) = \inf \max_i t_i \|a_i\|_{A_i},$$

or equivalently

$$\inf \max \frac{t_i \|a_i\|_{A_i}}{K_x(\vec{t}, a; \vec{A})} = 1.$$

But then also

$$\inf \max \left(\frac{t_i \|a_i\|_{A_i}}{K_x(\vec{t}, a; \vec{A})} \right)^{q_i} = 1.$$

Here we see that defining $\vec{s}$ by

$$s_i = t_i^{q_i} (K_x(\vec{t}, a; \vec{A}))^{1-q_i} \quad (i = 0, 1, \dots, n)$$

we obtain the equality

$$K_x(\vec{s}, a; \vec{A}[\vec{q}]) = K_x(\vec{t}, a; \vec{A}).$$

Up to function equivalences this is the same as (7.4) and (7.6). The statement about (7.5) and (7.6) is proved by the same method. $\ddagger$

Remark 7.1. We note that (7.4) (or (7.5)) describes a homeomorphism both on R_+^{n+1} and P_+^n .

Theorem 7.1. (The power theorem). Let $\vec{\eta} = q\vec{\theta}/\vec{q}$ with $1/q = \sum_{i=0}^n \theta_i/q_i$ (or $q = \sum_{i=0}^n \eta_i q_i$). Then holds

$$(\vec{A}_{\vec{\theta}, pq; K})[q] = (\vec{A}[\vec{q}])_{\vec{\eta} p; K}.$$

Proof: We are only interested in norm equivalences. In view of prop. 7.1 we can find differentiable functions $\kappa(\vec{t}, a; \vec{A})$ and $\kappa(\vec{s}, a; \vec{A}[\vec{q}])$, equivalent to $K(\vec{t}, a; \vec{A})$ and $K(\vec{s}, a; \vec{A}[\vec{q}])$ respectively. Let the differentiable mapping $\vec{s} = s(\vec{t})$ be defined by

$$s_i = t_i^{q_i} (\kappa(\vec{t}, a; \vec{A}))^{1-q_i} \quad (i = 0, 1, \dots, n).$$

Then by prop. 7.2

$$\kappa(\vec{t}, a; \vec{A}) \sim \kappa(\vec{s}, a; \vec{A}[\vec{q}]).$$

We define

$$\kappa_1(\vec{t}, a; \vec{A}) = \kappa(s(\vec{t}), a; \vec{A}[\vec{q}]).$$

Then κ_1 is a differentiable function, equivalent to $\kappa(\vec{t}, a; \vec{A})$ (and thus to $K(\vec{t}, a; \vec{A})$). We integrate over the level surfaces $\kappa_1(\vec{t}, a; \vec{A}) = 1$ and $\kappa(\vec{s}, a; \vec{A}[\vec{q}]) = 1$, which in fact are P-surfaces. Noting also that the measure $\omega(\vec{t}^{\vec{q}})$ is equivalent to $\omega(\vec{t})$, we obtain

$$\begin{aligned} \int (|\vec{s}^{-\vec{q}}| \kappa(\vec{s}, a; \vec{A}[\vec{q}]))^{pq} \omega(\vec{s}) &= \int_{\kappa(\vec{s}, a; \vec{A}[\vec{q}])=1} |\vec{s}^{-\vec{q}}|^{pq} \omega(\vec{s}) = \\ &= \int_{\kappa_1(\vec{t}, a; \vec{A})=1} |\vec{t}^{-\vec{q}}|^{pq} \omega(\vec{t}^{\vec{q}}) \sim \int_{\kappa_1(\vec{t}, a; \vec{A})=1} |\vec{t}^{-\vec{\theta}}|^{pq} \omega(\vec{t}) = \\ &= \int (|\vec{t}^{-\vec{\theta}}| \kappa_1(\vec{t}, a; \vec{A}))^{pq} \omega(\vec{t}). \quad \# \end{aligned}$$

8. Interpolation of L_p -spaces.

Throughout this section X is a measured space with a positive measure μ .

Let A be a Banach space and let w be a positive μ -measurable function, weight function, on X . By $L_q(w;A)$, or $L_q(w;A;X;\mu)$ when we want to be more specific, $1 \leq q \leq \infty$, we mean the Banach space of μ -measurable functions on X , corresponding to the norm

$$\|a\|_{L_q(w;A)} = \left(\int \|w(x)a(x)\|_A^q d\mu \right)^{1/q}$$

(with the usual interpretation if $q = \infty$). For the integration theory of functions taking values in Banach spaces, we refer to Lang [12], Ch. X,XI.

Now let $\vec{A} = \{A_0, A_1, \dots, A_n\}$ be a Banach tuple, $\vec{w} = (w_0, w_1, \dots, w_n)$ a tuple of weight functions on X and $\vec{q} = (q_0, q_1, \dots, q_n)$, $1 \leq q_i \leq \infty$ ($i = 0, 1, \dots, n$). We form the Banach tuple

$$L_{\vec{q}}(\vec{w};\vec{A}) = \{L_{q_0}(w_0;A_0), L_{q_1}(w_1;A_1), \dots, L_{q_n}(w_n;A_n)\}.$$

We agree to write $L_q(\vec{w};\vec{A})$ if $q_0 = q_1 = \dots = q_n = q$, $L_{\vec{q}}(w;\vec{A})$ if $w_0 = w_1 = \dots = w_n$, and $L_{\vec{q}}(\vec{w};A)$ if $A_0 = A_1 = \dots = A_n = A$.

The following theorem corresponds in the case $n = 1$ to a generalization of a classical theorem of M. Riesz. In the special case $q_0 = q_1 = \dots = q_n = 1$ it can be found in Foias-Lions [6]. It also has a J-analogue, th. 8.3 below.

Theorem 8.1. Let $\vec{\theta} \in H_+^n$. If $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$), then

$$(8.1) \quad (L_{\vec{q}}(\vec{w};\vec{A}))_{\vec{\theta};K} = L_1(|\vec{w}^{\vec{\theta}}|; (\vec{A}^{\vec{q}})_{\vec{\theta};K}).$$

If $1 \leq q_i \leq \infty$, ($i = 0, 1, \dots, n$), $1/q = \sum_{i=0}^n \theta_i/q_i$, then

$$(8.2) \quad (L_{\vec{q}}(\vec{w};\vec{A}))_{\vec{\theta};K} = L_q(|\vec{w}^{\vec{\theta}}|; \vec{A}_{\vec{\theta};K}).$$

(In (8.1) we have in fact equality of "norms", not only equivalence).

Restricting ourselves to the case $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$), following Peetre [22] (where $n = 1$), we prove (8.1). As a consequence of th. 7.1, we then obtain (8.2). The remaining cases can be treated for example by borrowing ideas from Peetre [22], Sec. 8 or Holmstedt [8], Sec. 5, "the parameter theorem".

The main step is to derive an expression for $K(\vec{t}, a; L_{\vec{q}}^{[\vec{q}]}(\vec{w}; \vec{A}))$ (lemma 8.4). This will be done in some detail, in a series of steps. One meets some difficulties with the measurability. To begin with, we will work with step functions: By an A -valued step function we mean a function of the form $\sum a^i \chi_{E_i}$ (finite sum), where, for any i , $a^i \in A$ and χ_{E_i} is the characteristic function of a set E_i of finite μ -measure. The set of all such functions will be denoted $\text{St}(A)$. By $\mathcal{M}(A)$ we denote the set of all A -valued μ -measurable functions (i.e. the functions which are pointwise limits a.e. of functions in $\text{St}(A)$). In accordance with our usual notation we write $\text{St}(\vec{A}) = \{\text{St}(A_0), \text{St}(A_1), \dots, \text{St}(A_n)\}$, $\mathcal{M}(\vec{A}) = \{\mathcal{M}(A_0), \mathcal{M}(A_1), \dots, \mathcal{M}(A_n)\}$.

Lemma 8.1. Let $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$) and let $a \in \Sigma(\mathcal{M}(\vec{A}))$. Then there exists a sequence $(\phi^k)_{k \in \mathbb{Z}_+} \subset \Sigma(\text{St}(\vec{A}))$ with the following properties:

$$(8.3) \quad K(\vec{t}, \vec{w}(\vec{x})^{\vec{q}}, a(x) - \phi^k(x); \vec{A}^{[\vec{q}]}) \rightarrow 0 \text{ a.e. as } k \rightarrow \infty,$$

if $a \in \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$ then

$$(8.4) \quad K(\vec{t}, a - \phi^k; L_{\vec{q}}^{[\vec{q}]}(\vec{w}; \vec{A})) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

if $a \notin \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$ then

$$(8.5) \quad K(\vec{t}, \phi^k; L_{\vec{q}}^{[\vec{q}]}(\vec{w}; \vec{A})) \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Proof: Let $a \in \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$, $a = a_0 + a_1 + \dots + a_n$ with $a_i \in L_{q_i}(w_i; A_i)$ ($i = 0, 1, \dots, n$). By fundamental properties of the integral (cf. eg. Lang [12], XI, 4), for each i ($i = 0, 1, \dots, n$) there exists a sequence $(\phi_i^k)_{k \in \mathbb{Z}_+} \subset \text{St}(A_i)$ such that

$$\phi_i^k(x) \rightarrow a_i(x) \quad \text{a.e.} \quad \text{as } k \rightarrow \infty,$$

$$\|\phi_i^k - a_i\|_{L_{q_i}(w_i; A_i)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

If we define $\phi^k = \phi_0^k + \phi_1^k + \dots + \phi_n^k$, then the sequence $(\phi^k)_{k \in \mathbb{Z}_+} \subset \Sigma(\text{St}(\vec{A}))$ has the asserted properties. The remaining

case $a \notin \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$ is treated in a similar manner. $\#$

In view of this lemma, in the calculation of the K-functional it suffices to consider step functions only. To take full advantage of this we also make the weight functions discrete valued. We have the obvious:

Lemma 8.2. Let $\epsilon > 0$. If w is defined by

$$w(x) = (1+\epsilon)^n \quad \text{if} \quad (1+\epsilon)^{n-1} < w(x) \leq (1+\epsilon)^n, \quad n \in \mathbb{Z},$$

then, for all $a \in L_q(w; A)$,

$$\|a\|_{L_q(w; A)} \leq \|a\|_{L_q(w; A)} \leq (1+\epsilon) \|a\|_{L_q(w; A)}. \quad \#$$

Lemma 8.3. Let $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$). If $a \in \Sigma(\text{St}(\vec{A}))$,

then

$$K(\vec{t}, a; L_{\vec{q}}(\vec{w}; \vec{A})) = \int K(\vec{t}w(x)^{\vec{q}}, a(x); \vec{A}^{\vec{q}}) d\mu.$$

Proof: We have

$$\begin{aligned} (8.6) \quad K(\vec{t}, a; L_{\vec{q}}(\vec{w}; \vec{A})) &= \\ &= \inf \int (t_0 w_0(x)^{q_0} \|a_0(x)\|_{A_0}^{q_0} + t_1 w_1(x)^{q_1} \|a_1(x)\|_{A_1}^{q_1} + \dots \\ &+ t_n w_n(x)^{q_n} \|a_n(x)\|_{A_n}^{q_n}) d\mu. \end{aligned}$$

In view of lemma 8.2 it suffices to consider discrete-valued weight functions. The step function a can then be expressed by a (not necessarily finite) sum $a = \sum a^j \chi_{E_j}$, where all the weight functions w_i ($i = 0, 1, \dots, n$) are constant on each of the sets E_j ($j = 0, 1, 2, \dots$). Thus, in (8.6), we may restrict ourselves to take infimum over functions $a_i = a_i(x)$ ($i = 0, 1, \dots, n$), which are constant on E_j ($j = 0, 1, 2, \dots$). But then "inf" and "f" commute, and the lemma is proved. $\#$

Actually, the same holds true for all $a \in \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$:

Lemma 8.4. Let $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$). If $a \in \Sigma(m(\vec{A}))$,

then

$$(8.7) \quad K(\vec{t}, a; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})) = \int K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x); \vec{A}[\vec{q}]) d\mu.$$

Here either both sides are finite and equal, or both are infinite.

Proof: Firstly, since a is pointwise limit of step functions, $K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x); \vec{A}[\vec{q}])$ is a pointwise limit of measurable functions, by (8.3). Thus it is measurable. Let $a \in \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$ and let $(\phi^k)_{k \in \mathbb{Z}_+}$ be the step functions of lemma 8.1. Clearly holds

$$\int K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x); \vec{A}[\vec{q}]) d\mu \leq K(\vec{t}, a; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})).$$

Application of this inequality to $a - \phi^k$ yields

$$\begin{aligned} & \left| \int K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x); \vec{A}[\vec{q}]) d\mu - \int K(\vec{t} \vec{w}(x)^{\vec{q}}, \phi^k(x); \vec{A}[\vec{q}]) d\mu \right| \leq \\ & \leq C \int K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x) - \phi^k(x); \vec{A}[\vec{q}]) d\mu \leq C K(\vec{t}, a - \phi^k; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})). \end{aligned}$$

From (8.4) then follows

$$\int K(\vec{t} \vec{w}(x)^{\vec{q}}, \phi^k(x); \vec{A}[\vec{q}]) d\mu \rightarrow \int K(\vec{t} \vec{w}(x)^{\vec{q}}, a(x); \vec{A}[\vec{q}]) d\mu \quad \text{as } k \rightarrow \infty.$$

But, also by means of (8.4),

$$K(\vec{t}, \phi^k; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})) \rightarrow K(\vec{t}, a; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})) \quad \text{as } k \rightarrow \infty.$$

In view of lemma 8.3, this proves (8.7) if $a \in \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$.

The case $a \notin \Sigma(L_{\vec{q}}(\vec{w}; \vec{A}))$ is treated in a similar manner. $\#$

Proof of th. 8.1. Lemma 8.4 and a change of the order of integration yields

$$\begin{aligned}
 ||a||_{(L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A}))_{\vec{n}_1; K}} &= \int |\vec{t}^{-\vec{n}}| K(\vec{t}, a; L_{\vec{q}}[\vec{q}](\vec{w}; \vec{A})) \omega(\vec{t}) = \\
 &= \int |\vec{t}^{-\vec{n}}| (\int K(\vec{t} \vec{w}(\vec{x})^{\vec{q}}, a(\vec{x}); \vec{A}[\vec{q}]) d\mu) \omega(\vec{t}) = \\
 &= \int (\int |\vec{t}^{-\vec{n}}| K(\vec{t} \vec{w}(\vec{x})^{\vec{q}}, a(\vec{x}); \vec{A}[\vec{q}]) \omega(\vec{t})) d\mu = \\
 &= \int |\vec{w}(\vec{x})^{\vec{n}\vec{q}}| ||a(\vec{x})||_{(\vec{A}[\vec{q}])_{\vec{n}_1; K}} d\mu .
 \end{aligned}$$

This proves (8.1). Applying th. 7.1, with $\vec{n}\vec{q}/q = \vec{\theta}$,

$$1/q = \sum_{i=0}^n \theta_i/q_i \quad (\text{or } q = \sum_{i=0}^n \eta_i q_i), \text{ we then obtain (8.2).} \quad \#$$

The next theorem deals with a special case of the more general situation of "field valued" functions. Generally, let $\{A(x)\}_{x \in X}$ be a field of Banach spaces attached to X . We denote by $L_q(w; \{A(x)\})$, $1 \leq q \leq \infty$, the space corresponding to the norm

$$||a||_{L_q(w; \{A(x)\})} = (\int ||w(x)a(x)||_{A(x)}^q d\mu)^{1/q} .$$

In general, dealing with such spaces will lead to considerable measurability difficulties. We restrict ourselves to the case of a discrete measured space X , where all functions are measurable. Then it is easy to modify the proof of th. 8.1 to cover also the following theorem (cf. Peetre [22] for the case $n = 1$).

Here by $\{\vec{A}(x)\}$ we mean the "field tuple" $\{A_0(x), A_1(x), \dots, A_n(x)\}_{x \in X}$.

Theorem 8.2. Let X be a discrete measured space. Let

$\vec{\theta} \in H_+^n$. If $1 \leq q_i < \infty$ ($i = 0, 1, \dots, n$), then

$$(8.8) \quad (L_{\vec{q}}[\vec{q}](\vec{w}; \{\vec{A}(x)\}))_{\vec{\theta}_1; K} = L_1(|\vec{w}^{\vec{\theta}\vec{q}}|; \{(\vec{A}(x)[\vec{q}])_{\vec{\theta}_1; K}\}) .$$

If $1 \leq q_i \leq \infty$ ($i = 0, 1, \dots, n$), $1/q = \sum_{i=0}^n \theta_i/q_i$, then

$$(8.9) \quad (L_{\vec{q}}(\vec{w}; \{\vec{A}(x)\}))_{\vec{\theta}_q; K} = L_q(|\vec{w}^{\vec{\theta}}|; \{\vec{A}(x)\}_{\vec{\theta}_q; K}) .$$

(In (8.8) we have in fact equality of "norms").

We also state, without proofs, the J-analogues of these theorems. In what follows, however, it will mostly be sufficient to know the K-results. We note that by means of the parameter theorem (cf. remark 4.6), our next theorem, th. 8.3, follows from Yoshikawa [25], prop. 3.1: $(L_{\vec{q}}(\vec{w}; \vec{A}))_{\vec{\theta q; J}} = L_q(|\vec{w}^{\vec{\theta}}|; \vec{A}_{\vec{\theta q; J}})$ if $1/q = \sum \theta_i/q_i$. (Cf. also Lions-Peetre [13] for the case $n = 1$). This proof can be modified to cover also the field valued case, th. 8.4.

Theorem 8.3. Let $\vec{\theta} \in H_+^n$, $1 \leq q_i \leq \infty$ ($i = 0, 1, \dots, n$) and let $1/q = \sum_{i=0}^n \theta_i/q_i$. Then holds

$$(L_{\vec{q}}(\vec{w}; \vec{A}))_{\vec{\theta q; J}} = L_q(|\vec{w}^{\vec{\theta}}|; \vec{A}_{\vec{\theta q; J}}).$$

Theorem 8.4. Let X be a discrete measured space. Let $\vec{\theta} \in H_+^n$, $1 \leq q_i \leq \infty$ ($i = 0, 1, \dots, n$) and let $1/q = \sum_{i=0}^n \theta_i/q_i$. Then holds

$$(L_{\vec{q}}(\vec{w}; \{\vec{A}(x)\}))_{\vec{\theta q; J}} = L_q(|\vec{w}^{\vec{\theta}}|; \{\vec{A}(x)_{\vec{\theta q; J}}\}).$$

Suppose that $\vec{A}_{\vec{\theta q; K}} = \vec{A}_{\vec{\theta q; J}}$, or in particular that $\mathcal{F}(\vec{A})$ is satisfied. If $1/q = \sum \theta_i/q_i$, then by th. 8.1 and th. 8.3,

$$(L_{\vec{q}}(\vec{w}; \vec{A}))_{\vec{\theta q; K}} = (L_{\vec{q}}(\vec{w}; \vec{A}))_{\vec{\theta q; J}} = L_q(|\vec{w}^{\vec{\theta}}|; \vec{A}_{\vec{\theta q}}).$$

At the time being we don't know whether the K- and J-spaces are equal also without the restriction on $\vec{\theta}$, $\vec{q}$ and q . In particular, we don't know whether in general the assumption that $\mathcal{F}(\vec{A})$ is satisfied implies that also $\mathcal{F}(L_{\vec{q}}(\vec{w}; \vec{A}))$ is satisfied. However, if $q_0 = q_1 = \dots = q_n = q$, so is the case:

Proposition 8.1. If $\mathcal{F}(\vec{A})$ is satisfied, then also $\mathcal{F}(L_q(\vec{w}; \vec{A}))$ is satisfied.

Proof: It is convenient to use, instead of K , the following functional:

$$K_q(\vec{t}, a; \vec{A}) = (K(\vec{t}^q, a; \vec{A}^{[q]}))^{1/q} = \\ = \inf((t_0 \|a_0\|_{A_0})^q + (t_1 \|a_1\|_{A_1})^q + \dots + (t_n \|a_n\|_{A_n})^q)^{1/q}.$$

Obviously, K_q is equivalent to K ($= K_1$). By lemma 8.4 holds

$$(8.10) \quad K_q(\vec{t}, a; L_q(\vec{w}; \vec{A})) = (\int K_q^q(\vec{t}\vec{w}(\vec{x}), a(x); \vec{A}) d\mu)^{1/q}.$$

Now, let $a \in \sigma(L_q(\vec{w}; \vec{A}))$. Then, by (8.10) and Minkowski's inequality,

$$\begin{aligned} & (\int (\int \frac{K_q(\vec{t}\vec{w}(\vec{x}), a(x); \vec{A})}{\max(\vec{t})} \omega(\vec{t})^q d\mu)^{1/q} \leq \\ & \leq \int (\int (\frac{K_q(\vec{t}\vec{w}(\vec{x}), a(x); \vec{A})}{\max(\vec{t})})^q d\mu)^{1/q} \omega(\vec{t}) = \\ & = \int \frac{K_q(\vec{t}, a; L_q(\vec{w}; \vec{A}))}{\max(\vec{t})} \omega(\vec{t}) < \infty. \end{aligned}$$

Thus, for almost every x , $a(x) \in \sigma(\vec{A})$. Since $\mathcal{F}(\vec{A})$ is satisfied we can find a function $v(\cdot, x) \in \mathcal{H}_0(\Delta(\vec{A}))$ such that

$$\begin{aligned} a(x) &= \int v(\vec{t}, x) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}), \\ J(\vec{t}, v(\vec{t}, x); \vec{A}) &\leq C K_q(\vec{t}, a(x); \vec{A}). \end{aligned}$$

By a step function argument we see that v can be so chosen that, for almost every $\vec{t}$, $v(\vec{t}, \cdot)$ is μ -measurable.

Set $u(\vec{t}, x) = v(\vec{t}\vec{w}(\vec{x}), x)$. Then, for almost every x , we have $u(\cdot, x) \in \mathcal{H}_0(\Delta(\vec{A}))$,

$$(8.11) \quad a(x) = \int u(\vec{t}, x) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A})$$

(a consequence of the invariance property (3.9)), and

$$(8.12) \quad J(\vec{t}\vec{w}(\vec{x}), u(\vec{t}, x); \vec{A}) \leq C K_q(\vec{t}\vec{w}(\vec{x}), a(x); \vec{A}).$$

Moreover, for almost every $\vec{t}$, $u(\vec{t}, \cdot)$ is μ -measurable. Now, rewriting (8.12) as

$$(t_i w_i(x) \|u(\vec{t}, x)\|_{A_i})^q \leq C K_q^q(\vec{t}\vec{w}(\vec{x}), a(x); \vec{A}) \quad (i = 0, 1, \dots, n),$$

an integration gives

$$t_i^q \int ||w_i(x)u(\vec{t},x)||_{A_i}^q d\mu \leq C \int K_q^q(\vec{t}w(\vec{x}),a(x);\vec{A})d\mu$$

(i = 0,1,...,n) .

In view of (8.10), this is the same as

$$J(\vec{t},u(\vec{t},\cdot);L_q(\vec{w};\vec{A})) \leq C K_q(\vec{t},a;L_q(\vec{w};\vec{A})).$$

But this inequality yields (cf. remark 5.3) that the integral $\int u(\vec{t},\cdot)\omega(\vec{t})$ converges in $\Sigma(L_q(\vec{w};\vec{A}))$. By (8.11) its value is a .

Hence $\mathcal{F}(L_q(\vec{w};\vec{A}))$ is satisfied. $\#$

It is easy to modify this proof to cover also the "field-valued" case:

Proposition 8.2. Let X be a discrete measured space. If $\mathcal{F}(\vec{A}(x))$ is satisfied for every $x \in X$ and $\sup_{x \in X} \gamma(\vec{A}(x)) < \infty$, then $\mathcal{F}(L_q(\vec{w};\{\vec{A}(x)\}))$ is satisfied.

(Here $\gamma(\vec{A})$ is the constant in the definition of $\mathcal{F}$, def. 5.2).

9. On the stability of K- and J-methods.

In this section we will study the following question: If we interpolate with respect to spaces, obtained by interpolation of $\vec{A}$, do we get a space which can be obtained directly from $\vec{A}$? Otherwise stated, we will study the question whether our class of interpolation functors is, in some sense, stable under compositions.

Our treatment is divided into three parts. In the first two, the question is whether, for suitable values of the parameters,

$$(\vec{A}_{\vec{\theta}^0 q_0}, \vec{A}_{\vec{\theta}^1 q_1}, \dots, \vec{A}_{\vec{\theta}^m q_m})_{\vec{\lambda} q} = \vec{A}_{\vec{\theta} q}.$$

In Sub-Section 9.1, in fact as a special case, we find that this is true for any $\vec{A}$, q and q_i ($i = 0, 1, \dots, m$), provided that the vectors $\vec{\theta}^i$ ($i = 0, 1, \dots, m$) not degenerate to some subspace of R^{n+1} . If they do, we have to put restrictions on $\vec{A}$, q and q_i ($i = 0, 1, \dots, m$). This case is treated in Sub-Section 9.2.

In the special case $n = 1$ we thus consider the interpolation space $(\vec{A}_{\vec{\theta}^0 q_0}, \vec{A}_{\vec{\theta}^1 q_1})_{\vec{\lambda} q}$. Sub-Section 9.1 then reduces to the case $\vec{\theta}^0 \neq \vec{\theta}^1$ and we obtain what is usually called the "reiteration theorem", see Peetre [16], Butzer-Berens [5]. Sub-Section 9.2 then corresponds to the degenerate case $\vec{\theta}^0 = \vec{\theta}^1$, see Peetre [17], Berenstein-Kerzman [2].

In Sub-Section 9.3 we prove a stability result of somewhat other kind, giving connections between interpolation functors of different orders. As a consequence we find that restricted to "well behaved" tuples, a n -functor can always be expressed in terms of 1-functors only.

Throughout this section, if not otherwise stated,

$\vec{A} = \{A_0, A_1, \dots, A_n\}$ is a Banach $(n+1)$ -tuple.

9.1. Stability results of type I.

Definition 9.1. Let $\vec{\theta} \in \bar{H}_+^n$. Let X be a Banach space such that

$$\Delta(\vec{A}) \subset X \subset \Sigma(\vec{A}).$$

We say that X is of class $\mathcal{C}_K(\vec{\theta}; \vec{A})$ if

$$(9.1) \quad K(\vec{t}, a; \vec{A}) \leq C |\vec{t}^{\vec{\theta}}| \|a\|_X, \quad a \in X,$$

that X is of class $\mathcal{C}_J(\vec{\theta}; \vec{A})$ if

$$(9.2) \quad \|a\|_X \leq C |\vec{t}^{-\vec{\theta}}| J(\vec{t}, a; \vec{A}), \quad a \in \Delta(\vec{A}),$$

and that X is of class $\mathcal{C}(\vec{\theta}; \vec{A})$ if it is both of class $\mathcal{C}_K(\vec{\theta}; \vec{A})$ and $\mathcal{C}_J(\vec{\theta}; \vec{A})$.

Remark 9.1. Obviously A_i is of class $\mathcal{C}(\vec{e}_i; \vec{A})$, $\vec{e}_i = i$:th unit vector ($i = 0, 1, \dots, n$). As a consequence of (4.3), (4.4), (4.10), (4.11), the spaces $\vec{A}_{\vec{\theta}q; K}$ and $\vec{A}_{\vec{\theta}q; J}$ are of class $\mathcal{C}(\vec{\theta}; \vec{A})$. These relations can also be used on interpolation spaces which do not involve all the spaces A_i ($i = 0, 1, \dots, n$). For example, $(A_0, A_1)_{(\theta_0, \theta_1)q}$ is of class $\mathcal{C}((\theta_0, \theta_1, 0, \dots, 0); \vec{A})$. We return to this situation in Remark 9.4 below.

The classes $\mathcal{C}_K(\vec{\theta}; \vec{A})$ and $\mathcal{C}_J(\vec{\theta}; \vec{A})$ can also be described by

Proposition 9.1. Let $\vec{\theta} \in \bar{H}_+^n$. Then:

i) X is of class $\mathcal{C}_K(\vec{\theta}; \vec{A})$ if and only if

$$(9.3) \quad X \subset \vec{A}_{\vec{\theta}\infty; K},$$

ii) X is of class $\mathcal{C}_J(\vec{\theta}; \vec{A})$ if and only if

$$(9.4) \quad \vec{A}_{\vec{\theta}1; J} \subset X,$$

iii) X is of class $\mathcal{C}(\vec{\theta}; \vec{A})$ if and only if

$$(9.5) \quad \vec{A}_{\vec{\theta}1; J} \subset X \subset \vec{A}_{\vec{\theta}\infty; K}.$$

1. The first part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x f(t) dt + \dots$$

where $f(x)$ is a function defined on the interval $[0, 1]$.

It is known that

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where $f(x)$ is a function defined on the interval $[0, 1]$.

It is known that

Proof: Obviously, iii) follows from i) and ii). The K-part being obvious, only the J-part need to be verified. That (9.4) implies (9.2) follows by means of (4.11):

$$\|a\|_{\vec{A}_{\vec{\theta}^1; J}} \leq C |\vec{t}^{-\vec{\theta}}| J(\vec{t}; a).$$

Conversely, suppose that (9.2) holds true. If $u \in \mathcal{H}_0(\Delta(\vec{A}))$, $a = \int u(\vec{t}) \omega(\vec{t})$, then

$$\|a\|_X \leq \int \|u(\vec{t})\|_X \omega(\vec{t}) \leq C \int |\vec{t}^{-\vec{\theta}}| J(\vec{t}, u(\vec{t})) \omega(\vec{t}).$$

Making u vary, we obtain (9.4). $\#\#$

Proposition 9.2. Let $\vec{\theta}^i \in \bar{H}_+^n$ ($i = 0, 1, \dots, m$), $\vec{\theta} \in H_+^n$, $\vec{\lambda} \in H_+^m$ with $\vec{\theta} = \sum_{i=0}^m \lambda_i \vec{\theta}^i$. Let $\vec{X} = \{X_0, X_1, \dots, X_m\}$ be a Banach tuple. Suppose that R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$.

i) If X_i is of class $\mathcal{C}_K(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, \dots, m$), then

$$\vec{X}_{\vec{\lambda}; K} \subset \vec{A}_{\vec{\theta}; K}.$$

ii) If X_i is of class $\mathcal{C}_J(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, \dots, m$), then

$$\vec{A}_{\vec{\theta}; J} \subset \vec{X}_{\vec{\lambda}; J}.$$

Proof: First we assume that $m = n$. The proof depends on prop. 3.2. We note that the vectors $\vec{\theta}^i$ ($i = 0, 1, \dots, n$) are linearly independent.

To begin with i), let $a = a_0 + a_1 + \dots + a_n$ with $a_i \in X_i$ ($i = 0, 1, \dots, n$). Then from (9.1), applied to X_i ($i = 0, 1, \dots, n$), we get

$$K(\vec{t}, a; \vec{A}) \leq C (|\vec{t}^{\vec{\theta}^0}| \|a_0\|_{X_0} + |\vec{t}^{\vec{\theta}^1}| \|a_1\|_{X_1} + \dots + |\vec{t}^{\vec{\theta}^n}| \|a_n\|_{X_n}).$$

Writing

$$(9.6) \quad \alpha(\vec{t}) = (|\vec{t}^{\vec{\theta}^0}|, |\vec{t}^{\vec{\theta}^1}|, \dots, |\vec{t}^{\vec{\theta}^n}|),$$

we therefore have

$$(9.7) \quad K(\vec{t}, a; \vec{A}) \leq C K(\alpha(\vec{t}), a; \vec{X}).$$

Application of $\Phi_{\vec{\theta}_q}$ to this inequality yields, in view of prop. 3.2,

$$\Phi_{\vec{\theta}_q} [K(\vec{t}, a; \vec{A})] \leq C \Phi_{\vec{\theta}_q} [K(\alpha(\vec{t}), a; \vec{X})] = C \Phi_{\vec{\lambda}_q} [K(\vec{t}, a; \vec{X})].$$

This is the desired inequality.

To prove (ii), let $a \in \vec{A}_{\vec{\theta}_q; J}$, $u \in \mathcal{H}_0(\Delta(\vec{A}))$ with

$$(9.8) \quad a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}),$$

$$(9.9) \quad \Phi_{\vec{\theta}_q} [J(\vec{t}, u(\vec{t}); \vec{A})] < \infty.$$

With α as in (9.6) and $D(\alpha)$ as in prop. 3.2, let v be defined by

$$(9.10) \quad D(\alpha)v(\alpha(\vec{t})) = u(\vec{t}).$$

Then from (9.2), applied to X_i ($i = 0, 1, \dots, n$), we get

$$(9.11) \quad D(\alpha)J(\alpha(\vec{t}), v(\alpha(\vec{t})); \vec{X}) = \max_{i=0,1,\dots,n} |\vec{t}^{\vec{\theta}_i}| ||u(\vec{t})||_{X_i} \leq \\ \leq C J(\vec{t}, u(\vec{t}); \vec{A}).$$

Application of $\Phi_{\vec{\theta}_q}$ yields, by means of prop. 3.2,

$$(9.12) \quad \Phi_{\vec{\lambda}_q} [J(\vec{t}, v(\vec{t}); \vec{X})] = D(\alpha)^{1/q} \Phi_{\vec{\theta}_q} [J(\alpha(\vec{t}), v(\alpha(\vec{t})); \vec{X})] \leq \\ \leq C \Phi_{\vec{\theta}_q} [J(\vec{t}, u(\vec{t}); \vec{A})].$$

Here the right-hand side is finite, by (9.9). Thus the integral $\int v(\vec{t}) \omega(\vec{t})$ is convergent in $\Sigma(\vec{X})$ (see remark 4.3). That its value is a follows from (3.21), in view of (9.8) and (9.10). We also have $v \in \mathcal{H}_0(\Delta(\vec{X}))$. Now, by variation of u and v in (9.12), the assertion is verified in the case $m = n$.

Next, let $m > n$. We "add" $m-n$ copies of A_n and form the $(m+1)$ -tuple

$$\vec{A}' = \{A_0, A_1, \dots, A_n, A_n, \dots, A_n\}.$$

By prop. 6.3 we have

$$\vec{A}'_{\vec{\eta}_q; K} = \vec{A}_{\vec{\theta}_q; K},$$

where $\vec{\theta} \in H_+^n$, $\vec{\eta} \in H_+^m$ and $\vec{\theta} = L(\vec{\eta}) = (\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n + \dots + \eta_m)$. By assumption, R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$. Then one can find $\vec{\eta}^i$ ($i = 0, 1, \dots, m$), linearly independent in R^{m+1} , such that $\vec{\theta}^i = L(\vec{\eta}^i)$ ($i = 0, 1, \dots, m$). If X_i is of class $\mathcal{C}_K(\vec{\theta}^i; \vec{A})$, then it is also of class $\mathcal{C}_K(\vec{\eta}^i; \vec{A}')$ ($i = 0, 1, \dots, m$). Now the above argument applies and yields

$$\vec{X}_{\vec{\lambda}_q; K} \subset \vec{A}_{\vec{\eta}_q; K}.$$

Using prop. 6.3 one more time, i) is proved. The J-case is treated in a similar way. $\#$

We now immediately get one of the main results of this section:

Theorem 9.1. Let $\vec{\theta}^i \in \bar{H}_+^n$ ($i = 0, 1, \dots, m$), $\vec{\theta} \in H_+^n$, $\vec{\lambda} \in H_+^m$ with $\vec{\theta} = \sum_{i=0}^m \lambda_i \vec{\theta}^i$. Let $\vec{X} = \{X_0, X_1, \dots, X_m\}$, where X_i is of class $\mathcal{C}(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, \dots, m$). If

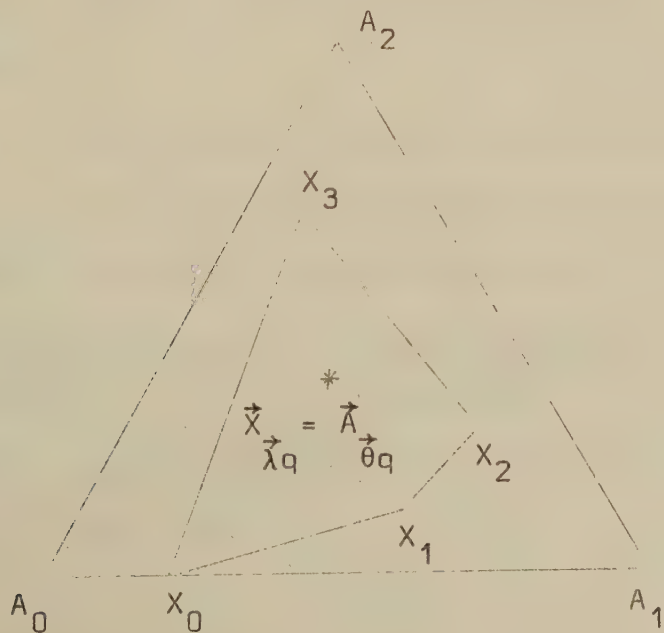
i) R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$,

ii) $\vec{A}_{\vec{\theta}_q; K} = \vec{A}_{\vec{\theta}_q; J}$ (or, in particular, $\mathcal{F}(\vec{A})$ is satisfied),

then

$$\vec{X}_{\vec{\lambda}_q} = \vec{A}_{\vec{\theta}_q}.$$

The theorem is illustrated in the following figure, where $n = 2$, $m = 3$. Here $\vec{\theta}$ and $\vec{\lambda}$ have the meanings of barycentric coordinates.



Proof: From prop. 9.1 and prop. 5.1 we get

$$\vec{A}_{\vec{\theta}_q;J} \subset \vec{X}_{\vec{\lambda}_q;J} \subset \vec{X}_{\vec{\lambda}_q;K} \subset \vec{A}_{\vec{\theta}_q;K}.$$

But here, by assumption, $\vec{A}_{\vec{\theta}_q;J} = \vec{A}_{\vec{\theta}_q;K}$. This proves the theorem. #

In view of remark 9.1, we have

Corollary 9.1. Let $\vec{\theta}^i \in H_+^n$ ($i = 0, 1, \dots, m$), $\vec{\theta} \in H_+^n$, $\vec{\lambda} \in H_+^m$ with $\vec{\theta} = \sum_{i=0}^m \lambda_i \vec{\theta}^i$. If

- i) R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$,
- ii) $\vec{A}_{\vec{\theta}^i q_i;K} = \vec{A}_{\vec{\theta}^i q_i;J}$ ($i = 0, 1, \dots, m$), $\vec{A}_{\vec{\theta}_q;K} = \vec{A}_{\vec{\theta}_q;J}$
(or, in particular, $\mathcal{F}(\vec{A})$ is satisfied),

then

$$(9.13) \quad (\vec{A}_{\vec{\theta}^0 q_0}, \vec{A}_{\vec{\theta}^1 q_1}, \dots, \vec{A}_{\vec{\theta}^m q_m})_{\vec{\lambda}_q} = \vec{A}_{\vec{\theta}_q}. \quad \#$$

For the case when R^{n+1} is not spanned by $\{\vec{\theta}^i\}_0^m$ we refer to th. 9.2 below.

The following proposition enables us to make repeated use of th. 9.1. In that context also prop. 9.4 below can be of interest.

Proposition 9.3. Let $\vec{\theta}^i \in \bar{H}_+^n$, $\vec{X} = \{X_0, X_1, \dots, X_m\}$ with X_i of class $\mathcal{C}(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, \dots, m$). Suppose that R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$. If $\mathcal{F}(\vec{A})$ is satisfied, then also $\mathcal{F}(\vec{X})$ is satisfied.

Proof: As in the proof of prop. 9.2, it suffices to consider the case $m = n$. Let $a \in \sigma(\vec{X})$. From (9.6), (9.7) and the fact that $\max(\alpha(\vec{t})) \leq \max(\vec{t})$, by means of prop. 3.2 we obtain

$$\int \frac{K(\vec{t}, a; \vec{A})}{\max(\vec{t})} \omega(\vec{t}) \leq C \int \frac{K(\alpha(\vec{t}), a; \vec{X})}{\max(\alpha(\vec{t}))} \omega(t) = C \int \frac{K(\vec{t}, a; \vec{X})}{\max(\vec{t})} \omega(t).$$

Thus $a \in \sigma(\vec{A})$. By assumption there exists a function

$u \in \mathcal{H}_0(\Delta(\vec{A}))$ such that

$$a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A}),$$

$$J(\vec{t}, u(\vec{t}); \vec{A}) \leq C K(\vec{t}, a; \vec{A}).$$

Let v be defined by (9.10). Obviously $v \in \mathcal{H}_0(\Delta(\vec{X}))$. By use of (9.7) and (9.11) we obtain, with $\vec{s} = \alpha(\vec{t})$,

$$\begin{aligned} J(\vec{s}, v(\vec{s}); \vec{X}) &\leq C J(\vec{t}, u(\vec{t}); \vec{A}) \leq \\ &\leq C K(\vec{t}, a; \vec{A}) \leq C K(\vec{s}, a; \vec{X}). \end{aligned}$$

Since $a \in \sigma(\vec{X})$ this inequality yields (cf. remark 5.3), that the integral $\int v(\vec{s}) \omega(\vec{s})$ is convergent in $\Sigma(\vec{X})$. In view of (9.8), (9.10) and (3.21), its value is a . Hence $\mathcal{F}(\vec{X})$ is satisfied. $\#$

Proposition 9.4. Let $\{\vec{\theta}^i\}_0^m$, $\vec{\theta}$, $\vec{\lambda}$ and $\vec{X}$ be as in th. 9.1. Suppose that R^{n+1} is spanned by $\{\vec{\theta}^i\}_0^m$ and $\mathcal{F}(\vec{A})$ is satisfied. Then a Banach space X is of class $\mathcal{C}(\vec{\theta}; \vec{A})$ if and only if it is of class $\mathcal{C}(\vec{\lambda}; \vec{X})$.

Proof: Let $m = n$. The "if" part follows from (9.7) and the relation (cf. (9.11))

$$J(\alpha(\vec{t}), a; \vec{X}) \leq J(\vec{t}, a; \vec{A}).$$

Conversely, if X is of class $\mathcal{C}(\vec{\theta}; \vec{A})$, then by prop. 9.1

$$\vec{A}_{\vec{\theta}_1} \subset X \subset \vec{A}_{\vec{\theta}_\infty}.$$

Thus, in view of th. 9.1,

$$\vec{X}_{\vec{\lambda}_1} \subset X \subset \vec{X}_{\vec{\lambda}_\infty}.$$

Using prop. 9.1 one more time, the proposition is verified. $\#$

Remark 9.2. The requirement on $\vec{\theta}^i$ ($i = 0, 1, \dots, m$) to span R^{n+1} is necessary above. To see this, we consider the Banach 4-tuple $\vec{A} = \{A_0, A_1, A_2, A_3\}$, where A_0, A_1 and A_2 are the spaces of remark 5.1 and $A_0 \cap A_3 = 0$. Let $\vec{\theta}^0 = (1, 0, 0, 0)$, $\vec{\theta}^1 = (0, 1, 0, 0)$ and $\vec{\theta}^2 = (0, 0, 1, 0)$, which vectors do not span R^4 . Then A_i is of class $\mathcal{C}(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, 2$). It is easily

seen that $\sigma(\vec{A}) = 0$, and thus $\mathcal{F}(\vec{A})$ is satisfied. But from remark 5.1 we know that $\mathcal{F}(\{A_0, A_1, A_2\})$ is not satisfied. Thus prop. 9.3 does not hold without the "span assumption". Clearly, this example shows that this assumption is necessary also in the other instances. (Concerning th. 9.1 (and cor. 9.1), a less peculiar counterexample is given in remark 10.1 below).

9.2 Stability results of type II:

With restrictions on $\vec{\lambda}, q$ and q_i instead of $\{\vec{\theta}^i\}_0^m$ in cor. 9.1, we have

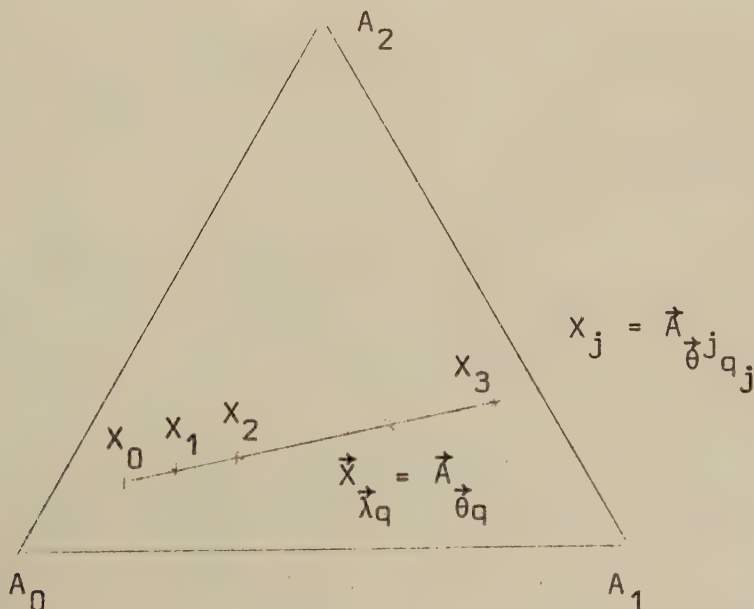
Theorem 9.2. Let $\vec{\theta}^i \in H_+^n$ ($i = 0, 1, \dots, m$), $\vec{\theta} \in H_+^n$, $\vec{\lambda} \in H_+^m$ with $\vec{\theta} = \sum_{i=0}^m \lambda_i \vec{\theta}^i$. If

- i) $1/q = \sum_{i=0}^m \lambda_i / q_i$,
- ii) $\vec{A}_{\vec{\theta}^i q_i; K} = \vec{A}_{\vec{\theta}^i q_i; J}$ ($i = 0, 1, \dots, m$), $\vec{A}_{\vec{\theta} q; K} = \vec{A}_{\vec{\theta} q; J}$
(or, in particular, $\mathcal{F}(\vec{A})$ is satisfied),

then

$$(9.14) \quad (\vec{A}_{\vec{\theta}^0 q_0}, \vec{A}_{\vec{\theta}^1 q_1}, \dots, \vec{A}_{\vec{\theta}^m q_m}) \vec{\lambda}_q = \vec{A}_{\vec{\theta} q}.$$

The theorem is illustrated in the following figure, where $n = 2$, $m = 3$.



Th. 9.2 is an immediate consequence of

Proposition 9.5. Let $\{\vec{\theta}^i\}_0^m$, $\vec{\theta}$, $\vec{\lambda}$ and q be as in th. 9.2.

Then

$$(9.15) \quad (\vec{A}_{\vec{\theta}^0_{q_0};K}, \vec{A}_{\vec{\theta}^1_{q_1};K}, \dots, \vec{A}_{\vec{\theta}^m_{q_m};K})_{\vec{\lambda}_{q;K}} \subset \vec{A}_{\vec{\theta}_{q;K}},$$

$$(9.16) \quad (\vec{A}_{\vec{\theta}^0_{q_0};J}, \vec{A}_{\vec{\theta}^1_{q_1};J}, \dots, \vec{A}_{\vec{\theta}^m_{q_m};J})_{\vec{\lambda}_{q;J}} \supset \vec{A}_{\vec{\theta}_{q;J}}.$$

Proof: We will use the theorems on field valued L_q -spaces, th. 8.2 and th. 8.4. Let $X = \{\vec{v} \in \mathbb{Z}^{n+1} | \min(\vec{v}) = 0\}$. We let all the L_q -spaces of this proof be assigned to X . Thus they consist of sequences defined on X . As norms on the K - and J -spaces we will use the discrete norms associated with X , see remark 4.5.

We set

$$\vec{A}_K = \{\vec{A}_{\vec{\theta}^0_{q_0};K}, \vec{A}_{\vec{\theta}^1_{q_1};K}, \dots, \vec{A}_{\vec{\theta}^m_{q_m};K}\},$$

$$\vec{A}_J = \{\vec{A}_{\vec{\theta}^0_{q_0};J}, \vec{A}_{\vec{\theta}^1_{q_1};J}, \dots, \vec{A}_{\vec{\theta}^m_{q_m};J}\},$$

$$\vec{L}(\Sigma) = \{L_{q_0}(|2^{-\vec{v}\vec{\theta}^0}|; \{\Sigma(2^{\vec{v}\vec{\lambda}}\vec{A})\}), L_{q_1}(|2^{-\vec{v}\vec{\theta}^1}|; \{\Sigma(2^{\vec{v}\vec{\lambda}}\vec{A})\}), \dots, L_{q_m}(|2^{-\vec{v}\vec{\theta}^m}|; \{\Sigma(2^{\vec{v}\vec{\lambda}}\vec{A})\})\},$$

$$\vec{L}(\Delta) = \{L_{q_0}(|2^{-\vec{v}\vec{\theta}^0}|; \{\Delta(2^{\vec{v}\vec{\lambda}}\vec{A})\}), L_{q_1}(|2^{-\vec{v}\vec{\theta}^1}|; \{\Delta(2^{\vec{v}\vec{\lambda}}\vec{A})\}), \dots, L_{q_m}(|2^{-\vec{v}\vec{\theta}^m}|; \{\Delta(2^{\vec{v}\vec{\lambda}}\vec{A})\})\}.$$

We have, in view of th. 8.2 and th. 8.4,

$$(9.17) \quad (\vec{L}(\Sigma))_{\vec{\lambda}_{q;K}} = L_q(|2^{-\vec{v}\vec{\theta}}|; \{\Sigma(2^{\vec{v}\vec{\lambda}}\vec{A})\}),$$

$$(9.18) \quad (\vec{L}(\Delta))_{\vec{\lambda}_{q;J}} = L_q(|2^{-\vec{v}\vec{\theta}}|; \{\Delta(2^{\vec{v}\vec{\lambda}}\vec{A})\}).$$

In verifying (9.5), let $\bar{a}$ be the constant sequence, equal to a . (That is, $\bar{a} = (a_{\vec{v}})_{\vec{v} \in X}$, with $a_{\vec{v}} = a$ for all $\vec{v} \in X$). Then

$$\|a\|_{\vec{A}_{\vec{\theta}; K}} = \|\bar{a}\|_{L_q(|2^{-\vec{v}\vec{\theta}}|; \{\Sigma(2^{\vec{v}}\vec{A})\})}.$$

Thus, by (9.17),

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}; K}} &= \|\bar{a}\|_{(\vec{L}(\Sigma))_{\vec{\lambda}; K}} = \\ &= \left(\sum_{\vec{v} \in X} (|2^{-\vec{v}\vec{\lambda}}|_{K(2^{\vec{v}}, \bar{a}; \vec{L}(\Sigma))})^q \right)^{1/q}. \end{aligned}$$

But here

$$K(2^{\vec{v}}, \bar{a}; \vec{L}(\Sigma)) \leq K(2^{\vec{v}}, a; \vec{A}_K).$$

since the right-hand side corresponds to taking infimum only over partitions $\bar{a} = \bar{a}_0 + \bar{a}_1 + \dots + \bar{a}_m$, where all sequences are constant. By this, (9.15) is verified.

Next, let $a \in \vec{A}_{\vec{\theta}; J}$. Then

$$\|a\|_{\vec{A}_{\vec{\theta}; J}} = \inf_u \|u\|_{L_q(|2^{-\vec{v}\vec{\theta}}|; \{\Delta(2^{\vec{v}}\vec{A})\})}.$$

Here the infimum is taken over all sequences $u = (u_{\vec{v}})_{\vec{v} \in X} \subset \Delta(\vec{A})$ such that

$$(9.19) \quad a = \sum_{\vec{v} \in X} u_{\vec{v}} \quad \text{in} \quad \Sigma(\vec{A}).$$

Thus, by (9.18),

$$(9.20) \quad \|a\|_{\vec{A}_{\vec{\theta}; J}} = \inf_u \|u\|_{(\vec{L}(\Delta))_{\vec{\lambda}; J}}.$$

In this expression

$$(9.21) \quad \|u\|_{(\vec{L}(\Delta))_{\vec{\lambda}; J}} = \inf_v \left(\sum_{\vec{\mu} \in X} (|2^{-\vec{\mu}\vec{\lambda}}|_{J(2^{\vec{\mu}}, v_{\vec{\mu}}; \vec{L}(\Delta))})^q \right)^{1/q}.$$

Here the infimum is taken over all sequences $v = (v_{\vec{\mu}})_{\vec{\mu} \in X} \subset \Delta(\vec{L}(\Delta))$, such that

$$(9.22) \quad u = \sum_{\vec{\mu} \in X} v_{\vec{\mu}} \quad \text{in} \quad \Sigma(\vec{L}(\Delta)).$$

For any $\vec{\mu} \in X$, we have $v_{\vec{\mu}} = (v_{\vec{\mu}\vec{v}})_{\vec{v} \in X} \in \Delta(\vec{A})$. From (9.19) and (9.22) then follows that

$$a = \sum_{\vec{\mu}, \vec{v} \in X} v_{\vec{\mu}\vec{v}} \quad \text{in} \quad \Sigma(\vec{A}).$$

Let $w = (w_{\vec{\mu}})_{\vec{\mu} \in X}$ be defined by

$$w_{\vec{\mu}} = \sum_{\vec{v} \in X} v_{\vec{\mu}\vec{v}}.$$

Then

$$a = \sum_{\vec{\mu} \in X} w_{\vec{\mu}} \quad \text{in} \quad \Sigma(\vec{A}).$$

Now it is quite easy to see that

$$(9.23) \quad J(2^{\vec{\mu}}, v_{\vec{\mu}}; \vec{L}(\Delta)) \geq J(2^{\vec{\mu}}, w_{\vec{\mu}}; \vec{A}_J).$$

Combination of (9.20), (9.21) and (9.23) yields

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}q;J}} &\geq \inf_{\vec{\mu} \in X} (|2^{-\vec{\mu}\vec{\lambda}}| J(2^{\vec{\mu}}, w_{\vec{\mu}}; \vec{A}_J))^q)^{1/q} \geq \\ &\geq \|a\|_{(\vec{A}_J)_{\vec{\lambda}q;J}}. \end{aligned}$$

This proves (9.16). (Note the rôle remark 4.3 played in the last inequality). $\#$

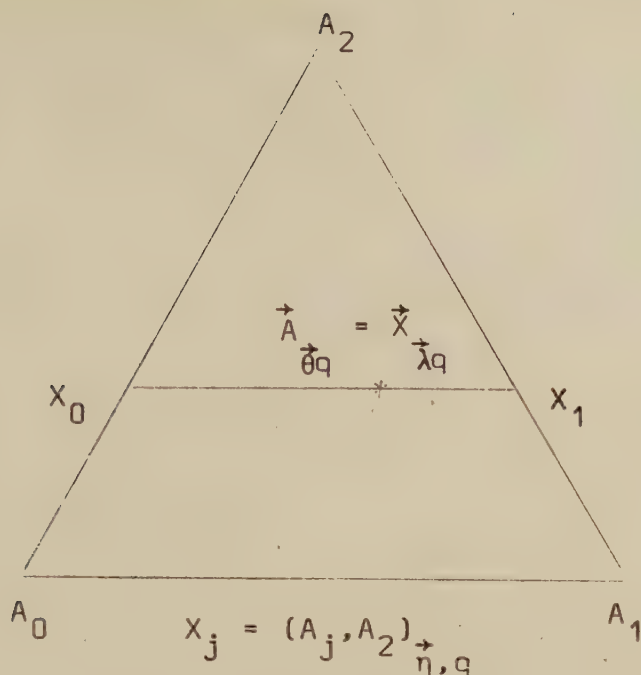
Below, in remark 10.1, we will give an example which shows that th. 9.2 is not true without the assumption $1/q = \sum \lambda_i/q_i$.

Omitting the assumption " $\mathbb{R}^{n+1}$ is spanned by $\{\vec{\theta}^i\}_0^m$ " in th. 9.1, we have:

Proposition 9.6. Let $\vec{\theta}^i \in H_+^n$ ($i = 0, 1, \dots, m$), $\vec{\theta} \in H_+^n$, $\vec{\lambda} \in H_+^m$ with $\vec{\theta} = \sum_{i=0}^m \lambda_i \vec{\theta}^i$. Let $\vec{X} = \{X_0, X_1, \dots, X_m\}$, where X_i is of class $G(\vec{\theta}^i; \vec{A})$ ($i = 0, 1, \dots, m$). Then

$$(9.24) \quad \vec{A}_{\vec{\theta}1;J} \subset \vec{X}_{\vec{\lambda}q;K} \subset \vec{A}_{\vec{\theta}\infty;K},$$

$$(9.25) \quad \vec{A}_{\vec{\theta}1;J} \subset \vec{X}_{\vec{\lambda}q;J} \subset \vec{A}_{\vec{\theta}\infty;K}.$$



Theorem 9.3. Let $1 \leq m \leq n - 1$. Let $\vec{\eta} = (\eta_0, \eta_{m+1}, \eta_{m+2}, \dots, \eta_n) \in H_+^{n-m}$, $\vec{\lambda} \in H_+^m$, and $\vec{\theta} \in H_+^n$ with

$$\begin{aligned} \theta_i &= \lambda_i \eta_0 & (i = 0, 1, \dots, m), \\ \theta_i &= \eta_i & (i = m+1, m+2, \dots, n). \end{aligned}$$

Let $\vec{B}^i = \{A_i, A_{m+1}, A_{m+2}, \dots, A_n\}$ ($i = 0, 1, \dots, m$). Suppose that

$$\vec{A}_{\vec{\theta}_q; K} = \vec{A}_{\vec{\theta}_q; J}, \quad \vec{B}^i_{\vec{\eta}_q; K} = \vec{B}^i_{\vec{\eta}_q; J} \quad (i = 0, 1, \dots, m)$$

(or, in particular, that $\mathcal{F}(\vec{A})$ and $\mathcal{F}(\vec{B}^i)$ ($i = 0, 1, \dots, m$) are satisfied). Then

$$(9.28) \quad (\vec{B}^0_{\vec{\eta}_q}, \vec{B}^1_{\vec{\eta}_q}, \dots, \vec{B}^m_{\vec{\eta}_q})_{\vec{\lambda}_q} = \vec{A}_{\vec{\theta}_q}.$$

The theorem is an immediate consequence of

Proposition 9.7. Let $\vec{\eta}$, $\vec{\lambda}$ and $\vec{\theta}$ be as in th. 9.3. Then

$$(9.29) \quad (\vec{B}^0_{\vec{\eta}_q; K}, \vec{B}^1_{\vec{\eta}_q; K}, \dots, \vec{B}^m_{\vec{\eta}_q; K})_{\vec{\lambda}_q; K} \subset \vec{A}_{\vec{\theta}_q; K},$$

$$(9.30) \quad (\vec{B}^0_{\vec{\eta}_q; J}, \vec{B}^1_{\vec{\eta}_q; J}, \dots, \vec{B}^m_{\vec{\eta}_q; J})_{\vec{\lambda}_q; J} \supset \vec{A}_{\vec{\theta}_q; J}.$$

Proof: First some notation. Let

$$\vec{s} = (s_0, s_1, \dots, s_m),$$

$$\vec{t} = (t_{m+1}, t_{m+2}, \dots, t_n).$$

Put

$$(\vec{s}, \vec{t}) = (s_0, s_1, \dots, s_m, t_{m+1}, \dots, t_n).$$

Let

$$\vec{s}' = (s_1, s_2, \dots, s_m).$$

We will use the measures

$$\lambda(\vec{s}') = \frac{ds_1}{s_1} \wedge \frac{ds_2}{s_2} \wedge \dots \wedge \frac{ds_m}{s_m},$$

$$\lambda(\vec{t}) = \frac{dt_{m+1}}{t_{m+1}} \wedge \frac{dt_{m+2}}{t_{m+2}} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

Put

$$\vec{A}' = \{A_{m+1}, A_{m+2}, \dots, A_n\},$$

$$\vec{B}^i = \{A_i, \vec{A}'\} \quad (i = 0, 1, \dots, m),$$

$$\vec{B}_K = \{\vec{B}_{\vec{\eta}_q; K}^0, \vec{B}_{\vec{\eta}_q; K}^1, \dots, \vec{B}_{\vec{\eta}_q; K}^m\},$$

$$\vec{B}_J = \{\vec{B}_{\vec{\eta}_q; J}^0, \vec{B}_{\vec{\eta}_q; J}^1, \dots, \vec{B}_{\vec{\eta}_q; J}^m\}.$$

Finally, let

$$\vec{\eta}' = (\eta_{m+1}, \eta_{m+2}, \dots, \eta_n),$$

$$\vec{\lambda}' = (\lambda_1, \lambda_2, \dots, \lambda_m).$$

Now to the proof. We start with (9.29). Instead of K we use the equivalent functional

$$K_q(\vec{t}, a; \vec{A}) = \inf((t_0 \|a_0\|_{A_0})^q + (t_1 \|a_1\|_{A_1})^q + \dots + (t_n \|a_n\|_{A_n})^q)^{1/q}.$$

$$\text{Let } a \in (\vec{B}_K)_{\vec{\lambda}_q; K} = ((A_0, \vec{A}')_{\vec{\eta}_q; K}, (A_1, \vec{A}')_{\vec{\eta}_q; K}, \dots, (A_m, \vec{A}')_{\vec{\eta}_q; K})_{\vec{\lambda}_q; K}.$$

Here

$$(A_i, \vec{A}')_{\vec{n}_q; K} = (s_i^{-1+1/\eta_0} A_i, s_i^{-1} \vec{A}')_{\vec{n}_q; K} \quad (i = 0, 1, \dots, m).$$

Then

$$\begin{aligned} K_q^q(\vec{s}, a; \vec{B}_K) &= \inf_{i=0}^m s_i^q ||a_i||^q (s_i^{-1+1/\eta_0} A_i, s_i^{-1} \vec{A}')_{\vec{n}_q; K} = \\ &= \inf_{i=0}^m \int (|\vec{t}^{-\vec{n}'}| s_i K_q((1, \vec{t}), a; \{s_i^{-1+1/\eta_0} A_i, s_i^{-1} \vec{A}'\}))^q \lambda(\vec{t}) \\ &\geq \int (|\vec{t}^{-\vec{n}'}| K_q((\vec{s}^{1/\eta_0}, \vec{t}), a; \vec{A}))^q \lambda(\vec{t}). \end{aligned}$$

By integration in $\vec{s}$ over the P-surface $s_0 = 1$, on which $\omega(\vec{s}) = \lambda(\vec{s}')$, we obtain

$$\begin{aligned} ||a||_{(\vec{B}_K)_{\vec{\lambda}_q; K}}^q &= \int (|\vec{s}'^{-\vec{\lambda}'}| K_q(\vec{s}, a; \vec{B}_K))^q \lambda(\vec{s}') \geq \\ &\geq \int (|\vec{s}'^{-\vec{\lambda}'}| |\vec{t}^{-\vec{n}'}| K_q((1, \vec{s}'^{1/\eta_0}, \vec{t}), a; \vec{A}))^q \lambda(\vec{s}') \wedge \lambda(\vec{t}) = \\ &= c \int (|\vec{s}'^{-\eta_0 \vec{\lambda}'}| |\vec{t}^{-\vec{n}'}| K_q((1, \vec{s}', \vec{t}), a; \vec{A}))^q \lambda(\vec{s}') \wedge \lambda(\vec{t}) = \\ &= c ||a||_{\vec{A}_{\vec{\theta}_q; K}}^q. \end{aligned}$$

This proves (9.29).

Turning to (9.30), let $a \in \vec{A}_{\vec{\theta}_q; J}$, $u \in \mathcal{H}_0(\Delta(\vec{A}))$ with

$$a = \int u(\vec{s}, \vec{t}) \omega(\vec{s}, \vec{t}) \quad \text{in } \Sigma(\vec{A}).$$

Then

$$J((\vec{s}, \vec{t}), u(\vec{s}, \vec{t}); \vec{A}) = \max_{i=0, 1, \dots, m} J((s_i, \vec{t}), u(\vec{s}, \vec{t}); \vec{B}^i).$$

An integration in $\vec{t}$ yields

$$\begin{aligned} (9.31) \quad &\int (|\vec{t}^{-\vec{n}'}| J((\vec{s}, \vec{t}), u(\vec{s}, \vec{t}); \vec{A}))^q \lambda(\vec{t}) \geq \\ &\geq \max_{i=0, 1, \dots, m} \int (|\vec{t}^{-\vec{n}'}| J((s_i, \vec{t}), u(\vec{s}, \vec{t}); \vec{B}^i))^q \lambda(\vec{t}). \end{aligned}$$

From now on, let $s_0 = 1$. Set

$$\eta_0^{-m} v(1, \vec{s}') = \int u(1, \vec{s}', {}^{1/\eta_0} \vec{t}) \lambda(\vec{t}) .$$

The reason for having η_0 here will soon be clear. If we extend v to be a 0-homogeneous function in the variable $\vec{s}$, then

$$a = \int v(\vec{s}) \omega(\vec{s}) \quad \text{in} \quad \Sigma(\vec{A}).$$

From (9.31) we now obtain

$$\begin{aligned} & (f(|\vec{t}^{-\eta}| J((1, \vec{s}', \vec{t}), u(1, \vec{s}', \vec{t}); \vec{A}))^q \lambda(\vec{t}))^{1/q} \geq \\ & \geq C \max(|v(1, \vec{s}', \eta_0)|_{\vec{B}_{\vec{\eta}q; J}^0}, s_1^{\eta_0} |v(1, \vec{s}', \eta_0)|_{\vec{B}_{\vec{\eta}q; J}^1}, \dots \\ & s_m^{\eta_0} |v(1, \vec{s}', \eta_0)|_{\vec{B}_{\vec{\eta}q; J}^m}) = C J((1, \vec{s}', \eta_0), v(1, \vec{s}', \eta_0); \vec{B}_J). \end{aligned}$$

Then

$$\begin{aligned} & \int (|\vec{s}', -\eta_0 \vec{\lambda}'| \cdot |\vec{t}^{-\eta}| J((1, \vec{s}', \vec{t}), u(1, \vec{s}', \vec{t}); \vec{A}))^q \lambda(\vec{s}') \wedge \lambda(\vec{t}) \geq \\ & \geq C \int (|\vec{s}', -\eta_0 \vec{\lambda}'| J((1, \vec{s}', \eta_0), v(1, \vec{s}', \eta_0); \vec{B}_J))^q \lambda(\vec{s}') = \\ & = C \int (|\vec{s}', -\vec{\lambda}'| J((1, \vec{s}'), v(1, \vec{s}'); \vec{B}_J))^q \lambda(\vec{s}'). \end{aligned}$$

Taking infimum here, also having remark 4.3 in mind, (9.30) is proved. $\#$

Remark 9.3. Th. 9.3 can probably be improved, allowing in (9.28) different q_i 's, obeying a relation $\sum \lambda_i / q_i = 1/q$. For the K-part this can be proved by means of the power theorem, th. 7.1. If $m = 1$ this also follows from Yoshikawa [25], prop. 4.1, 4.2, by means of the parameter theorem (cf. remark 4.6).

Remark 9.4. Let $\vec{A} \in \mathcal{B}_n$. Consider the interpolation spaces with respect to $\vec{A}$, formed by means of compositions of K- and J-functors of order $< n$. In particular we have the spaces of remark 4.7, formed by means of 1-functors only. Looking at

th. 9.3, one may ask whether there is any general connection between these spaces and $\vec{A}_{\vec{\theta}q;K}$, $\vec{A}_{\vec{\theta}q;J}$. In general there are no equalities. However, so much is clear, that any space X of this kind is of class $\mathcal{G}(\vec{\theta};\vec{A})$ for some $\vec{\theta}$. Thus, in view of prop. 9.1,

$$\vec{A}_{\vec{\theta}1;J} \subset X \subset \vec{A}_{\vec{\theta}\infty;K}.$$

We also observe the possibility to use th. 9.1 in certain cases.

10. Applications to sequence spaces and to spaces of Besov- and Sobolev-type.

The plan of this section is as follows: In Sub-Section 10.1 we define and state some interpolation properties of a class of sequence spaces. By means of these spaces, in Sub-Section 10.2 we define Besov spaces. These turn out to be retracts of the sequence spaces, and thus inherit their interpolation properties. In Sub-Section 10.3 we define Sobolev spaces and prove some interpolation results for them, mainly by use of the Besov spaces and the results of Section 9. Finally, in Sub-Section 10.4 we list some imbedding properties of the Besov and Sobolev spaces.

10.1. The spaces $S^{\vec{r}q}_\ell$.

Definition 10.1. Let $r \in R$, $1 \leq q \leq \infty$. By $\ell^{rq}(A)$ we mean the set of all sequences $a = \{a_v\}_{v \in \bar{Z}_+}$ in A , such that

$$(10.1) \quad \|a\|_{\ell^{rq}(A)} = \left(\sum_{v \in \bar{Z}_+} (2^{vr} \|a_v\|_A)^q \right)^{1/q} < \infty$$

(with the usual interpretation if $q = \infty$).

One readily verifies that (10.1) is a norm, under which $\ell^{rq}(A)$ is a Banach space.

By iteration, we get

Definition 10.2. Let $\vec{r} = (r_1, r_2, \dots, r_d) \in R^d$, $1 \leq q \leq \infty$.

Then we define

$$S^{\vec{r}q}_\ell(A) = \ell^{r_1q}(\ell^{r_2q}(\dots(\ell^{r_dq}(A))\dots)).$$

Thus a sequence $\{a_{\vec{v}}\}_{\vec{v} \in \bar{Z}_+^d}$ belongs to $S^{\vec{r}q}_\ell(A)$ if and only if

$$(10.2) \quad \|a\|_{S^{\vec{r}q}_\ell(A)} = \left(\sum_{\vec{v} \in \bar{Z}_+^d} (2^{\vec{v}\vec{r}} \|a_{\vec{v}}\|_A)^q \right)^{1/q} < \infty.$$

Here (10.2) defines a norm, under which $S^{\vec{r}q}_\ell(A)$ is a Banach space.

The following interpolation result is fundamental in what follows (cf. Peetre [20] for the case $n = d = 1$):

Lemma 10.1. Let $\vec{u}^i = c_i \vec{e}^i$ where $0 \neq c_i \in \mathbb{R}$, $\vec{e}^i =$ the i :th unit vector in $\mathbb{R}^d$ ($i = 1, 2, \dots, d$), $\vec{e}^0 = (0, 0, \dots, 0)$. Let $\vec{\theta} \in H_+^d$, $1 \leq q \leq \infty$ and let $\vec{u} = \sum_{i=0}^d \theta_i \vec{u}^i$. Then

$$(10.3) \quad (S_{\vec{u}^0}^{0,1} \ell(A), S_{\vec{u}^1}^{1,1} \ell(A), \dots, S_{\vec{u}^d}^{d,1} \ell(A))_{\vec{\theta} q; K} = S_{\vec{u}}^{q,1} \ell(A).$$

Proof: Let $\vec{c} = (c_1, c_2, \dots, c_d)$ and $\vec{\gamma} = 2^{\vec{c}}$. Let $a = \{a_{\vec{v}}\}_{\vec{v} \in \bar{Z}_+^d}$. Then

$$\|a\|_{S_{\vec{u}^i}^{i,1} \ell} = \sum_{\vec{v} \in \bar{Z}_+^d} |2^{c_i \vec{v} \cdot \vec{e}^i} a_{\vec{v}}| = \sum_{\vec{v} \in \bar{Z}_+^d} |\vec{\gamma}^{\vec{v} \cdot \vec{e}^i} a_{\vec{v}}| \quad (i=0, 1, \dots, d)$$

Put $\vec{\theta}' = (\theta_1, \theta_2, \dots, \theta_d)$. Since $\vec{u} = \sum_{i=1}^d c_i \theta_i \vec{e}^i = \vec{c} \vec{\theta}'$, we have

$$(10.4) \quad \|a\|_{S_{\vec{u}}^{q,1} \ell} = \left(\sum_{\vec{v} \in \bar{Z}_+^d} (|2^{\vec{v} \cdot \vec{u}}| |a_{\vec{v}}|)^q \right)^{1/q} = \left(\sum_{\vec{v} \in \bar{Z}_+^d} (|\vec{\gamma}^{\vec{v} \cdot \vec{\theta}'}| |a_{\vec{v}}|)^q \right)^{1/q}$$

For the interpolation space in (10.3), we will use the (equivalent) discrete norm (see remark 4.5)

$$(10.5) \quad \left(\sum_{\vec{\mu} \in \mathbb{Z}^d} (|\vec{\gamma}^{-\vec{\mu} \cdot \vec{\theta}'}| K((1, \vec{\gamma}^{\vec{\mu}}), a))^q \right)^{1/q}.$$

We have to study $K((1, \vec{\gamma}^{\vec{\mu}}), a)$. Lemma 8.4 yields

$$\begin{aligned} K((1, \vec{\gamma}^{\vec{\mu}}), a) &= K((1, \vec{\gamma}^{\vec{\mu}}), a; \{S_{\vec{u}^0}^{0,1} \ell(A), S_{\vec{u}^1}^{1,1} \ell(A), \dots, S_{\vec{u}^d}^{d,1} \ell(A)\}) = \\ &= \sum_{\vec{v} \in \bar{Z}_+^d} K((1, \vec{\gamma}^{\vec{\mu} + \vec{v}}), a_{\vec{v}}; A) = \sum_{\vec{v} \in \bar{Z}_+^d} \min(1, \vec{\gamma}^{\vec{\mu} + \vec{v}}) \|a_{\vec{v}}\|_A. \end{aligned}$$

Here we see

$$(10.6) \quad \|a_{\vec{v}}\|_A \leq K((1, \vec{\gamma}^{-\vec{v}}), a),$$

$$(10.7) \quad K((1, \vec{\gamma}^{\vec{\mu}}), a) = \sum_{\vec{v} \geq \vec{\mu}} \min(1, \vec{\gamma}^{\vec{v}}) \|a_{\vec{v} - \vec{\mu}}\|_A.$$

We are now able to prove the equivalence of the norms (10.4) and (10.5). From (10.6) follows immediately

$$(10.8) \quad \left(\sum_{\vec{v} \in \bar{Z}_+^d} (|\vec{\gamma}^{-\vec{v}\vec{\theta}}, |||a_{\vec{v}}|||_A)^q \right)^{1/q} \leq \left(\sum_{\vec{v} \in Z^d} (|\vec{\gamma}^{-\vec{v}\vec{\theta}}, |K((1, \vec{\gamma}^{\vec{v}}), a))|^q \right)^{1/q}.$$

On the other hand, by means of (10.7) and Minkowski's inequality, we get

$$\begin{aligned} (10.9) \quad & \left(\sum_{\vec{\mu} \in Z^d} (|\vec{\gamma}^{-\vec{\mu}\vec{\theta}}, |K((1, \vec{\gamma}^{\vec{\mu}}), a))|^q \right)^{1/q} = \\ & = \left(\sum_{\vec{\mu} \in Z^d} (|\vec{\gamma}^{-\vec{\mu}\vec{\theta}}, | \sum_{\vec{v} \geq \vec{\mu}} \min(1, \vec{\gamma}^{\vec{v}}) |||a_{\vec{v}-\vec{\mu}}|||_A)^q \right)^{1/q} \leq \\ & \leq \sum_{\vec{v} \in Z^d} \min(1, \vec{\gamma}^{\vec{v}}) \left(\sum_{\vec{\mu} \leq \vec{v}} (|\vec{\gamma}^{-\vec{\mu}\vec{\theta}}, |||a_{\vec{v}-\vec{\mu}}|||_A)^q \right)^{1/q} = \\ & = \left(\sum_{\vec{v} \in Z^d} \min(1, \vec{\gamma}^{\vec{v}}) |\vec{\gamma}^{-\vec{v}\vec{\theta}}, | \right) \left(\sum_{\vec{\mu} \in \bar{Z}_+^d} (|\vec{\gamma}^{\vec{\mu}\vec{\theta}}, |||a_{\vec{\mu}}|||_A)^q \right)^{1/q}. \end{aligned}$$

Here, by a discrete version of (3.20), the first factor is finite.

By this, the lemma is proved. $\#$

In what follows we will often make the assumption that " R^d is affinely spanned by $\{\vec{r}^i\}_0^n$ ". We recall that this means that the linear space R^d is spanned by the vectors $\vec{r}^i - \vec{r}^0$ ($i = 1, 2, \dots, n$). The concept of (linear) span will also appear. As in Sub-Section 9.1, this happens in connection with some set $\{\vec{\theta}^i\}$ of interpolation parameters.

Theorem 10.1. Let $\vec{\theta} \in H_+^n$, $\vec{r}^i \in R^d$, $1 \leq q$, $q_i \leq \infty$ ($i=0, 1, \dots, n$) and let $\vec{r} = \sum_{i=0}^n \theta_i \vec{r}^i$.

i) If R^d is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$(10.10) \quad (S_{\vec{r}^0 q_0 \ell(A)}, S_{\vec{r}^1 q_1 \ell(A)}, \dots, S_{\vec{r}^n q_n \ell(A)})_{\vec{\theta} q} = S_{\vec{r} q \ell(A)}.$$

ii) If $1/q = \sum_{i=0}^n \theta_i / q_i$, then

$$(10.11) \quad (S_{\vec{r}^0 q_0 \ell(A_0)}, S_{\vec{r}^1 q_1 \ell(A_1)}, \dots, S_{\vec{r}^n q_n \ell(A_n)})_{\vec{\theta} q; K} = S_{\vec{r} q \ell(\vec{A}_{\vec{\theta} q; K})},$$

$$(10.12) \quad (S_{\vec{r}^0 q_0 \ell(A_0)}, S_{\vec{r}^1 q_1 \ell(A_1)}, \dots, S_{\vec{r}^n q_n \ell(A_n)})_{\vec{\theta} q; J} = S_{\vec{r} q \ell(\vec{A}_{\vec{\theta} q; J})}.$$

Proof: Since ii) is an immediate consequence of th. 8.1 and th. 8.3, only i) needs a proof. First let $\vec{r}^i \neq 0$ ($i = 0, 1, \dots, n$). We are going to use the reiteration theorem, th. 9.1 (or cor. 9.1), together with lemma 10.1. To this end we choose c_i ($i = 0, 1, \dots, d$) so that $\vec{r}^i$ ($i = 0, 1, \dots, n$) belong to the open convex hull of $\vec{u}^i = c_i \vec{e}^i$ ($i = 0, 1, \dots, d$). Then, by (10.3),

$$(10.13) \quad S^{\vec{r}^i q_i}_{\ell} = (S^{\vec{u}^0}_{\ell}, S^{\vec{u}^1}_{\ell}, \dots, S^{\vec{u}^d}_{\ell})_{\vec{\theta}^i q_i; K}$$

for some $\vec{\theta}^i \in H_+^d$ ($i = 0, 1, \dots, n$). Under our assumptions, R^{d+1} is (linearly) spanned by $\{\vec{\theta}^i\}_0^n$. Prop. 8.1 yields that $\mathcal{F}(\{S^{\vec{u}^0}_{\ell}, S^{\vec{u}^1}_{\ell}, \dots, S^{\vec{u}^d}_{\ell}\})$ is satisfied. Hence th. 9.1 (or cor. 9.1) applies. We get

$$(S^{\vec{r}^0 q_0}_{\ell}, S^{\vec{r}^1 q_1}_{\ell}, \dots, S^{\vec{r}^n q_n}_{\ell})_{\vec{\theta} q} = (S^{\vec{u}^0}_{\ell}, S^{\vec{u}^1}_{\ell}, \dots, S^{\vec{u}^d}_{\ell})_{\vec{\lambda} q}$$

where $\vec{r} = \sum_{i=0}^n \theta_i \vec{r}^i = \sum_{i=0}^d \lambda_i \vec{u}^i$. Using lemma 10.1 one more time,

(10.10) is proved in the case $\vec{r}^i \neq 0$ ($i = 0, 1, \dots, n$). After a simple modification, this argument can also be used in the remaining cases. $\#$

In particular, if $A_0 = A_1 = \dots = A_n = A$ in th. 10.1, we note that (10.11), (10.12) take care of some of the cases when (10.10) fails (and vice versa).

Concerning the condition $\mathcal{F}$, we have:

Proposition 10.1. Let $\vec{r}^i \in R^d$, $1 \leq q$, $q_i \leq \infty$ ($i = 0, 1, \dots, n$).

i) If R^d is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$\{S^{\vec{r}^0 q_0}_{\ell(A)}, S^{\vec{r}^1 q_1}_{\ell(A)}, \dots, S^{\vec{r}^n q_n}_{\ell(A)}\}$$

satisfies condition $\mathcal{F}$.

ii) If $\vec{A} = \{A_0, A_1, \dots, A_n\}$ satisfies condition $\mathcal{F}$, so does also

$$\{S^{\vec{r}^0 q_0}_{\ell(A_0)}, S^{\vec{r}^1 q_1}_{\ell(A_1)}, \dots, S^{\vec{r}^n q_n}_{\ell(A_n)}\}.$$

Proof: By means of prop. 9.3, part i) comes out from the proof of (10.10) in th. 10.1. Part ii) is an immediate consequence of prop. 8.1. $\#$

Remark 10.1. The following example shows the necessity of the "span assumption" in th. 9.1 and of the assumption $1/q = \sum_{i=0}^n \lambda_i/q_i$ in th. 9.2. At the same time it also shows the necessity of the corresponding assumptions in th. 10.1. We consider the triple

$$\vec{A} = \{S^{(0,0)q_\ell}, S^{(1,0)q_\ell}, S^{(0,1)q_\ell}\}.$$

Then $\mathcal{F}(\vec{A})$ is satisfied, by prop. 10.1. Let $\vec{\theta}^0 = (\theta_0^0, \theta_1^0, \theta_2^0)$, $\vec{\theta}^1 = (\theta_0^1, \theta_1^1, \theta_2^1)$ (which vectors do not span R^3). In view of (10.10), we have

$$\begin{aligned} \{\vec{A}_{\vec{\theta}^0 q}, \vec{A}_{\vec{\theta}^1 q}\} &= \{S^{(\theta_1^0, \theta_2^0)q_\ell}, S^{(\theta_1^1, \theta_2^1)q_\ell}\} \\ &= \{\ell^{\theta_1^0 q} (\ell^{\theta_2^0 q}), \ell^{\theta_1^1 q} (\ell^{\theta_2^1 q})\}. \end{aligned}$$

Let $\theta_2^0 = \theta_2^1 = \theta_2$, $\theta_1^0 \neq \theta_1^1$. Application of (10.10) with $d = 1$ yields

$$(10.14) \quad (\vec{A}_{\vec{\theta}^0 q}, \vec{A}_{\vec{\theta}^1 q})_{\vec{\lambda} q} = \ell^{\theta_1^1 q'} (\ell^{\theta_2 q}), \quad \theta_1 = \lambda_0 \theta_1^0 + \lambda_1 \theta_1^1.$$

But by (10.10) with $d = 2$, $\vec{\theta} = (\theta_0, \theta_1, \theta_2) = \lambda_0 \vec{\theta}^0 + \lambda_1 \vec{\theta}^1$ we have

$$(10.15) \quad \vec{A}_{\vec{\theta} q} = S^{(\theta_1, \theta_2)q'}_\ell = \ell^{\theta_1 q'} (\ell^{\theta_2 q'}).$$

If $q \neq q'$, (10.14) and (10.15) are not equal. This proves the assertion.

Remark 10.2. The situation of lemma 10.1 can also be described by saying that in the following sense the class of all spaces $S^{\vec{r} q}_\ell$, $\vec{r} \in R^d$, is of dimension d under interpolation.

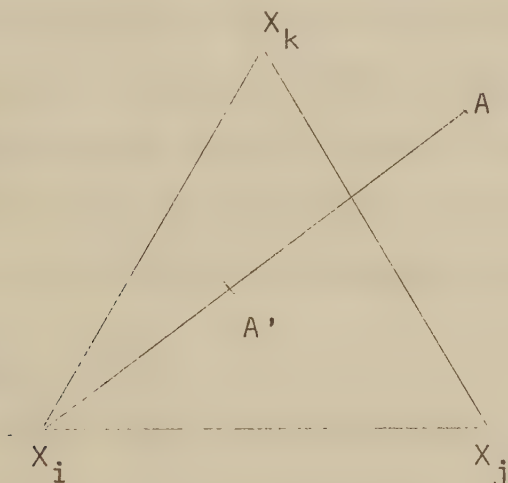
To begin with, we say that $\vec{X} = \{X_0, X_1, \dots, X_d\}$ is independent (under interpolation) if none of the spaces X_i ($i = 0, 1, \dots, d$)

is of class $\mathcal{C}(\vec{\theta}, \cdot)$, $\vec{\theta} \in \mathbb{R}_+^{d-1}$, with respect to the remaining. If $\vec{X}$ is not independent, we say it is dependent (under interpolation).

Next, let $\mathcal{S}$ be a class of Banach spaces, all continuously imbedded in some space A . Let $X_i \in \mathcal{S}$ ($i = 0, 1, \dots, d$), $\vec{X} = \{X_0, X_1, \dots, X_d\}$. We say that $\mathcal{S}$ is spanned (under interpolation) by $\vec{X}$, if to every $A \in \mathcal{S}$ there exists $A' \in \mathcal{S}$ such that

- i) $\{X_i, A, A'\}$ is dependent for some i ,
- ii) A' is of class $\mathcal{C}(\vec{\theta}; \vec{X})$.

In figure:



If here $\vec{X}$ is independent, we say that $\mathcal{S}$ is of dimension d (under interpolation). It is readily verified, by means of prop. 9.4, that this number is well-defined if it exists.

Thus, for example, the class of all $S_{\vec{r}q}^{\vec{r}q}$, $\vec{r} \in \mathbb{R}^d$, is of dimension d (lemma 10.1). We will see that this is the case also for the spaces we are going to define next, the spaces $S_p^{\vec{r}qB}$, $\vec{r} \in \mathbb{R}^d$ (th. 10.2 below). For w and A fixed, the class of all $L_q(w; A)$, $1 \leq q \leq \infty$, is of dimension 1 (by th. 6.1). We also see that nothing is attained by using interpolation of higher order than the dimension of the class in question.

Remark 10.3. We return to remark 4.7. Consider the triple $\{S^{(0,0)1}_\ell, S^{(1,0)1}_\ell, S^{(0,1)1}_\ell\}$. We have (lemma 10.1)

$$(S^{(0,0)1}_\ell, S^{(1,0)1}_\ell, S^{(0,1)1}_\ell)_{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)_q} = S^{\left(\frac{1}{3}, \frac{1}{3}\right)_q}_\ell,$$

for any $q, 1 \leq q \leq \infty$. Then, in view of th. 9.3,

$$\begin{aligned} & ((S^{(0,0)1}_\ell, S^{(1,0)1}_\ell)_{\left(\frac{1}{3}, \frac{2}{3}\right)_q}, (S^{(0,0)1}_\ell, S^{(0,1)1}_\ell)_{\left(\frac{1}{3}, \frac{2}{3}\right)_q})_{\left(\frac{1}{2}, \frac{1}{2}\right)_q} = \\ & = S^{\left(\frac{1}{3}, \frac{1}{3}\right)_q}_\ell. \end{aligned}$$

This relation, or any other of the kind discussed in remark 4.7, is not quite easy to prove only by use of 1-functors.

Trying to do this, sooner or later it seems necessary to introduce (some analogue of) the K-functor of order 2. The difficulties one meets even in this simple case motivate the introduction of K- and J-functors of general orders.

10.2. The spaces $S^{\vec{r}q}_p B$.

Following Peetre [20], we define Besov spaces by means of the sequence spaces of the preceding sub-section.

To this end, let $\{\phi_k\}_0^\infty$ be a sequence of C^∞ -functions on R^N , such that

$$\sum_{k=0}^\infty \hat{\phi}_k(\xi) = 1,$$

$$\hat{\phi}_k(\xi) = \hat{\phi}(\xi/2^k) \quad (k = 1, 2, \dots),$$

$$\text{supp } \hat{\phi} = \{\xi \mid 1/2 \leq ||\xi|| \leq 2\}.$$

(Here $||\xi|| = (\xi_1^2 + \xi_2^2 + \dots + \xi_N^2)^{1/2}$). We then have

$$\text{supp } \hat{\phi}_0 = \{\xi \mid ||\xi|| \leq 2\},$$

$$\text{supp } \hat{\phi}_k = \{\xi \mid 2^{k-1} \leq ||\xi|| \leq 2^{k+1}\}.$$

Let $\mathcal{S}'(R^N; A)$ be the space of all tempered distributions on R^N with values in a Banach space A .

Definition 10.3. (Besov spaces). By $B_p^{rq}(R^N; A)$ we mean the set of all $f \in \mathcal{F}'(R^N; A)$ for which

$$\{\phi_k \times f\}_{k \in \bar{Z}_+} \in \ell^{rq}(L_p(A)).$$

Whenever possible, we write $B_p^{rq}(A)$ or B_p^{rq} instead of $B_p^{rq}(R^N; A)$.

A norm on $B_p^{rq}(A)$ is defined by

$$\begin{aligned} f \mapsto \|f\|_{B_p^{rq}(A)} &= \|\{\phi_k \times f\}_{k \in \bar{Z}_+}\|_{\ell^{rq}(L_p(A))} = \\ &= \left(\sum_{k \in \bar{Z}_+} (2^{kr} \|\phi_k \times f\|_{L_p(A)})^q \right)^{1/q}. \end{aligned}$$

Under this norm, $B_p^{rq}(A)$ is a Banach space.

Remark 10.4. If $r > 0$ the space $B_p^{rq}(R^N; A)$ can also be characterized in terms of the modulus of continuity. This was done in the original definition, cf. e.g. Besov [3]. That our functions are vectorvalued causes no problem. Let $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x} + \vec{h}) - f(\vec{x})$ and let m be an integer, $m > r$. We define "the m :th modulus of continuity" by

$$\omega^m(t; f) = \sup_{\|\vec{h}\| \leq t} \|\Delta_{\vec{h}}^m f\|_{L_p(A)}.$$

Then the space $B_p^{rq}(R^N; A)$ can be characterized by

$$\|f\|_{L_p(A)} + \left(\int_0^\infty (t^{-r} \omega^m(t; f))^q \frac{dt}{t} \right)^{1/q} < \infty,$$

in the sense of norm equivalence. Concerning the equivalence of the definitions, we refer to Peetre [18], [20].

Following Nikolskij [15] (cf. also Grisvard [7] and Amanov [1]), by iteration we get

Definition 10.4. Let $\vec{N} = (N_1, N_2, \dots, N_d) \in Z_+^d$,

$\vec{r} = (r_1, r_2, \dots, r_d) \in R^d$, $1 \leq p, q \leq \infty$. We define

$$S_p^{\vec{r}q} B(R^{\vec{N}}; A) = B_p^{r_1 q}(R^{N_1}; B_p^{r_2 q}(\dots (B_p^{r_d q}(R^{N_d}; A)) \dots)).$$

Dropping the domains, we write shortly

$$S_p^{\vec{r}q} B(A) = B_p^{r_1q} (B_p^{r_2q} (\dots (B_p^{r_dq} (A)) \dots)).$$

On $S_p^{\vec{r}q} B(A)$, the norm is thus defined in the following way:

Let $(\phi_{k_i}^{(i)})_{k_i \in \bar{Z}_+}$ ($i = 0, 1, \dots, d$) correspond to the i :th step in def. 10.4, $\vec{k} = (k_1, k_2, \dots, k_d)$. Put $\phi_{\vec{k}} = \phi_{k_1}^{(1)} \otimes \phi_{k_2}^{(2)} \otimes \dots \otimes \phi_{k_d}^{(d)}$ (direct product). Then

$$||f||_{S_p^{\vec{r}q} B(A)} = \left(\sum_{\vec{k} \in \bar{Z}_+^d} (|2^{\vec{k}\vec{r}}| ||\phi_{\vec{k}} \times f||_{L_p(A)})^q \right)^{1/q}.$$

Remark 10.5. If $\vec{r} \in Z_+^d$ the space $S_p^{\vec{r}q} B$ can also be characterized by means of certain moduli of continuity (cf. Amanov [1]).

This follows by repeated use of remark 10.4. To this end, let

ω_i be the modulus of continuity, corresponding to the domain R^{N_i} ($i = 1, 2, \dots, d$). Let $\vec{n} = (n_1, n_2, \dots, n_d) \in \bar{Z}_+^d$. Recursively we define

$$\omega^{\vec{n}}(\vec{t}; f) = \omega_1^{n_1}(t_1; \omega_2^{n_2}(t_2; \dots; (\omega_d^{n_d}(t_d; f)) \dots)).$$

Here we make the agreement $\omega^0(t; f) = ||f||_{L_p}$. Thus, if $n_i = 0$

for some i , then, in fact, $\omega^{\vec{n}}(\vec{t}; f)$ does not depend on t_i .

Now let $\vec{m} \in Z_+^d$, $\vec{m} > \vec{r}$, and let $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_d)$ where $\epsilon_i = 0$ or 1 ($i = 1, 2, \dots, d$). Let $i_1, i_2, \dots, i_v$ be the indices for which $\epsilon_i = 1$. We define

$$\lambda_{\vec{\epsilon}}^{\vec{r}}(\vec{t}) = \frac{dt_{i_1}}{t_{i_1}} \frac{dt_{i_2}}{t_{i_2}} \dots \frac{dt_{i_d}}{t_{i_d}}.$$

Then $S_p^{\vec{r}q} B$ is characterized by

$$\sum_{\vec{\epsilon}} \left(\int (|t^{\vec{\epsilon}\vec{r}}| \omega^{\vec{\epsilon}\vec{m}}(\vec{t}; f))^q \lambda_{\vec{\epsilon}}^{\vec{r}}(\vec{t}) \right)^{1/q} < \infty. \quad \#$$

Many properties of the spaces $S_p^{\vec{r}q} B$ are inherited from

the spaces $B_p^{r_1q}$ under the iterative procedure, by which they are defined. For example we get several inclusions, see Sub-Section 10.4.

In this way we also prove the following lemma. It enables us to use the results about retracts in Section 5. Doing so, the interpolation properties of the spaces $S_p^{\vec{r}q}B$ will be derived from those of the spaces $S^{\vec{r}q}_\ell$.

Lemma 10.2. $S_p^{\vec{r}q}B(A)$ is a retract of $S^{\vec{r}q}_\ell(L_p(A))$. Moreover, in the diagram

$$\begin{array}{ccc} S_p^{\vec{r}q}B(A) & \xrightarrow{\text{id}} & S_p^{\vec{r}q}B(A) \\ I \searrow & & \nearrow P \\ & S^{\vec{r}q}_\ell(L_p(A)) & \end{array}$$

the mappings P and I can be chosen independently of $p, q, \vec{r}$ and A .

Proof: We consider first the case $d = 1$. Let ψ be a C^∞ -function such that

$$\begin{aligned} \hat{\psi}(\vec{\xi}) &= 1 \quad \text{if} \quad 1/2 \leq ||\vec{\xi}|| \leq 2, \\ \hat{\psi}(\vec{\xi}) &= 0 \quad \text{outside} \quad 1/4 \leq ||\vec{\xi}|| \leq 4. \end{aligned}$$

A sequence $\{\psi_k\}_{k=0}^\infty$ is defined by

$$\begin{aligned} \hat{\psi}_0(\vec{\xi}) &= 1 \quad \text{if} \quad ||\vec{\xi}|| \leq 2, \quad \hat{\psi}_0(\vec{\xi}) = 0 \quad \text{if} \quad ||\vec{\xi}|| \geq 4, \\ \hat{\psi}_k(\vec{\xi}) &= \hat{\psi}(\vec{\xi}/2^k) \quad (k = 1, 2, \dots). \end{aligned}$$

The definition is motivated by

$$(10.16) \quad \phi_k \times \psi_k = \phi_k \quad (k = 0, 1, \dots).$$

This relation follows immediately by means of a Fourier transformation.

Let

$$P : \{f_k\}_{k=0}^\infty \longmapsto \sum_{k=0}^\infty \psi_k \times f_k,$$

$$I : f \longmapsto \{\phi_k \times f\}_{k=0}^\infty,$$

where f and f_k ($k = 0, 1, \dots$) are vectorvalued tempered distribution. Obviously

$$I : B_p^{\vec{r}q}(A) \rightarrow \ell^{\vec{r}q}(L_p(A)).$$

For P we have

$$\begin{aligned} ||P(\{f_k\}_{k=0}^{\infty})||_{B_p^{rq}(A)} &= \\ &= \left(\sum_{n=0}^{\infty} (2^{nr} || \sum_{k=0}^{\infty} \phi_n \times \psi_k \times f_k ||_{L_p(A)})^q \right)^{1/q}. \end{aligned}$$

Here, if $n > 2$, $\phi_n \times \psi_k \neq 0$ only if $n-2 \leq k \leq n+2$. From Young's inequality and the fact that $||\phi_n||_{L_1} = ||\phi||_{L_1}$,

$||\psi_k||_{L_1} = ||\psi||_{L_1}$, we get

$$\begin{aligned} || \sum_{k=0}^{\infty} \phi_n \times \psi_k \times f_k ||_{L_p} &= || \sum_{k=n-2}^{n+2} \phi_n \times \psi_k \times f_k ||_{L_p} \leq \\ &\leq \sum_{k=n-2}^{n+2} ||\phi_n||_{L_1} ||\psi_k||_{L_1} ||f_k||_{L_p} = C ||f_k||_{L_p}. \end{aligned}$$

The remaining terms $n = 0, 1, 2$ are estimated in the same way.

Thus

$$||P(\{f_k\}_{k=0}^{\infty})||_{B_p^{rq}(A)} \leq C ||\{f_k\}||_{\ell^{rq}(L_p(A))},$$

i.e.

$$P : \ell^{rq}(L_p(A)) \rightarrow B_p^{rq}(A).$$

Finally, in view of (10.16) holds

$$(P \circ I)(f) = \sum_{k=0}^{\infty} \psi_k \times \phi_k \times f = \sum_{k=0}^{\infty} \phi_k \times f = \delta \times f = f$$

where δ is the Dirac measure. This proves the lemma in the case $d = 1$. (Note the independence of p, q, r and A).

For d general, let I_j and P_j correspond to the j :th step in forming $S_p^{\vec{r}q}B(A)$ and $S_p^{\vec{r}q}\ell(A)$ ($j = 1, 2, \dots, d$). Then the mappings $I = I_1 \otimes I_2 \otimes \dots \otimes I_d$ and $P = P_1 \otimes P_2 \otimes \dots \otimes P_d$ (direct products) have the prescribed properties. $\#$

In the following theorem, part iii), with $n = 1$, can be found in Grisvard [7] (cf. also Peetre [20], where in addition $d = 1$):

Theorem 10.2. Let $\vec{\theta} \in H_+^n$, $1 \leq p, p_i, q, q_i \leq \infty$, $\vec{r}^i \in R^d$ ($i = 0, 1, \dots, n$) and let $\vec{r} = \sum_{i=0}^n \theta_i \vec{r}^i$.

i) If R^d is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$(10.17) \quad (S_p^{\vec{r}^0 q_0 B(A)}, S_p^{\vec{r}^1 q_1 B(A)}, \dots, S_p^{\vec{r}^n q_n B(A)})_{\vec{\theta} q} = S_p^{\vec{r} q B(A)}.$$

ii) If $1/q = \sum_{i=0}^n \theta_i / q_i$, then

$$(10.18) \quad (S_p^{\vec{r}^0 q_0 B(A)}, S_p^{\vec{r}^1 q_1 B(A)}, \dots, S_p^{\vec{r}^n q_n B(A)})_{\vec{\theta} q} = S_p^{\vec{r} q B(A)}.$$

iii) If $1/q = \sum_{i=0}^n \theta_i / q_i = \sum_{i=0}^n \theta_i / p_i$, then

$$(10.19) \quad (S_{p_0}^{\vec{r}^0 q_0 B(A_0)}, S_{p_1}^{\vec{r}^1 q_1 B(A_1)}, \dots, S_{p_n}^{\vec{r}^n q_n B(A_n)})_{\vec{\theta} q; K} = S_q^{\vec{r} q B(\vec{A}_{\vec{\theta} q; K})},$$

$$(10.20) \quad (S_{p_0}^{\vec{r}^0 q_0 B(A_0)}, S_{p_1}^{\vec{r}^1 q_1 B(A_1)}, \dots, S_{p_n}^{\vec{r}^n q_n B(A_n)})_{\vec{\theta} q; J} = S_q^{\vec{r} q B(\vec{A}_{\vec{\theta} q; J})}.$$

Proof: To fix the ideas, we prove (10.17). In view of

lemma 10.2, we have a diagram

$$\begin{array}{ccc} \{S_p^{\vec{r}^0 q_0 B}, S_p^{\vec{r}^1 q_1 B}, \dots, S_p^{\vec{r}^n q_n B}\} & \xrightarrow{\text{id}} & \{S_p^{\vec{r}^0 q_0 B}, S_p^{\vec{r}^1 q_1 B}, \dots, S_p^{\vec{r}^n q_n B}\} \\ & \searrow I \quad \nearrow P & \\ & \{S_p^{\vec{r}^0 q_0 \ell(L_p)}, S_p^{\vec{r}^1 q_1 \ell(L_p)}, \dots, S_p^{\vec{r}^n q_n \ell(L_p)}\} & \end{array}$$

Prop. 5.3 then yields

$$\begin{array}{ccc} (S_p^{\vec{r}^0 q_0 B}, S_p^{\vec{r}^1 q_1 B}, \dots, S_p^{\vec{r}^n q_n B})_{\vec{\theta} q} & \xrightarrow{\text{id}} & (S_p^{\vec{r}^0 q_0 B}, S_p^{\vec{r}^1 q_1 B}, \dots, S_p^{\vec{r}^n q_n B})_{\vec{\theta} q} \\ & \searrow I \quad \nearrow P & \\ & (S_p^{\vec{r}^0 q_0 \ell(L_p)}, S_p^{\vec{r}^1 q_1 \ell(L_p)}, \dots, S_p^{\vec{r}^n q_n \ell(L_p)})_{\vec{\theta} q} & \end{array}$$

Here the subscripts K and J can be omitted in view of

prop. 10.1 and prop. 5.2. By (10.10) we have

$$(S_p^{\vec{r}^0 q_0 \ell(L_p)}, S_p^{\vec{r}^1 q_1 \ell(L_p)}, \dots, S_p^{\vec{r}^n q_n \ell(L_p)})_{\vec{\theta} q} = S_p^{\vec{r} q \ell(L_p)}.$$

Comparing the above diagram with that of lemma 10.2, we obtain (10.17). The remaining cases (10.18) - (10.20) are proved in the same way. $\#$

Concerning condition $\mathcal{F}$, we have:

Proposition 10.2. Let $\vec{r}^i \in \mathbb{R}^d$, $1 \leq q_i \leq \infty$ ($i = 0, 1, \dots, n$).

i) If $\mathbb{R}^d$ is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$\{S_p^{\vec{r}^0 q_0} B(A), S_p^{\vec{r}^1 q_1} B(A), \dots, S_p^{\vec{r}^n q_n} B(A)\}$$

satisfies the condition $\mathcal{F}$.

ii) If $\vec{A} = \{A_0, A_1, \dots, A_n\}$ satisfies the condition $\mathcal{F}$, so does also

$$\{S_p^{\vec{r}^0 q_0} B(A_0), S_p^{\vec{r}^1 q_1} B(A_1), \dots, S_p^{\vec{r}^n q_n} B(A_n)\}.$$

Proof: Immediate consequences of prop. 5.2 and lemma 10.2, in combination with prop. 10.1. $\#$

10.3. The spaces $S_p^r L$.

Definition 10.5. (Sobolev spaces). By $L_p^r(\mathbb{R}^N; A)$ we mean the set of all $f \in \mathcal{S}'(\mathbb{R}^N; A)$ for which

$$J^r f = \mathcal{F}^{-1}((1 + \|\xi\|^2)^{r/2} \mathcal{F}f) \in L_p(A).$$

Whenever possible we write $L_p^r(A)$ or L_p^r instead of $L_p^r(\mathbb{R}^N; A)$.

A norm on $L_p^r(A)$ is defined by

$$f \mapsto \|f\|_{L_p^r(A)} = \|J^r f\|_{L_p(A)}.$$

Under this norm, $L_p^r(A)$ is a Banach space.

Remark 10.6. If $1 < p < \infty$ and r positive, L_p^r has also the following characterization: $f \in L_p^r$ if and only if f and all its partial derivatives of order $\leq r$ belong to L_p . (Here the derivative is interpreted in distribution sense). This equivalence is a consequence of Michlin's multiplier theorem, see e.g. Hörmander [9].

Let $\mathcal{A}$ be a $\mathcal{C}^*$ -algebra and $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space $\mathcal{H}$.

For any $\mathcal{C}^*$ -algebra $\mathcal{A}$, we define the $\mathcal{K}$ -tensor product $\mathcal{A} \otimes \mathcal{K}$ as the completion of the algebraic tensor product $\mathcal{A} \otimes \mathcal{K}$ with respect to the norm

$$\| \sum_{i=1}^n a_i \otimes e_{ii} \|_{\mathcal{A} \otimes \mathcal{K}} = \left(\sum_{i=1}^n \|a_i\|_{\mathcal{A}}^2 \right)^{1/2},$$
 where $\{e_{ii}\}$ is a sequence of mutually orthogonal rank one projections in $\mathcal{K}$.

It is well known that $\mathcal{A} \otimes \mathcal{K}$ is isomorphic to the multiplier algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ of the $\mathcal{C}^*$ -module $\mathcal{A} \otimes \mathcal{K}$ over $\mathcal{A} \otimes \mathcal{K}$.

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 where $\{e_{ii}\}$ is a sequence of mutually orthogonal rank one projections in $\mathcal{K}$.

It is well known that $\mathcal{A} \otimes \mathcal{K}$ is isomorphic to the multiplier algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ of the $\mathcal{C}^*$ -module $\mathcal{A} \otimes \mathcal{K}$ over $\mathcal{A} \otimes \mathcal{K}$.

Definition 10.6. Let $\vec{N} = (N_1, N_2, \dots, N_d) \in \mathbb{Z}_+^d$,
 $\vec{r} = (r_1, r_2, \dots, r_d) \in \mathbb{R}^d$. We define

$$S_p^{\vec{r}} L(\vec{N}; A) = L_p^{r_1}(R^{N_1}; L_p^{r_2}(\dots(L_p^{r_d}(R^{N_d}; A))\dots)).$$

Dropping the domains, we write shortly

$$S_p^{\vec{r}} L(A) = L_p^{r_1}(L_p^{r_2}(\dots(L_p^{r_d}(A))\dots)).$$

On $S_p^{\vec{r}} L(A)$, the norm is thus defined in the following way:
 Let $J_i^{r_i}$ correspond to the i :th step in def. 10.6. Set $J^{\vec{r}} =$
 $= J_1^{r_1} \otimes J_2^{r_2} \otimes \dots \otimes J_d^{r_d}$ (direct product). Then

$$\|f\|_{S_p^{\vec{r}} L(A)} = \|J^{\vec{r}} f\|_{L_p(A)}.$$

Remark 10.7. If $1 < p < \infty$, $\vec{r} \in \bar{\mathbb{Z}}_+^d$, the space $S_p^{\vec{r}} L$ can
 also be characterized by means of derivatives. This follows by
 repeated use of remark 10.6. For instance, if $N_1 = N_2 = \dots =$
 $= N_d = 1$, $f \in S_p^{\vec{r}} L$ if and only if all partial derivatives
 $\vec{D}^{\vec{s}} f \in L_p$, $0 \leq \vec{s} \leq \vec{r}$. (Here $\vec{D} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d})$).

The plan is now to show that $S_p^{\vec{r}} L$ is of class $\mathcal{C}(\vec{\theta}, \vec{B})$
 for some $\vec{\theta}$, where $\vec{B}$ is a tuple of Besov spaces. Then it
 is possible to apply the results of Sub-Section 9.1, in com-
 bination with those of Sub-Section 10.2.

The following lemma is fundamental

Lemma 10.3. We have the isomorphisms

$$(10.22) \quad J^{\vec{\alpha}} : S_p^{\vec{r}} L \rightarrow S_p^{\vec{r}-\vec{\alpha}} L,$$

$$(10.23) \quad J^{\vec{\alpha}} : S_p^{\vec{r}q} B \rightarrow S_p^{\vec{r}-\vec{\alpha}q} B.$$

Proof: It suffices to consider the case $d = 1$. Obviously,
 $J^{\vec{\alpha}}$ has the inverse $J^{-\vec{\alpha}}$. Thus, if $J^{\vec{\alpha}}$ obeys (10.22), (10.23),
 then it is automatically an isomorphism.

We prove (10.23), (10.22) being obvious. Let $f \in B_p^{rq}$. Then

$$(10.24) \quad ||J^\alpha f||_{B_p^{r-\alpha q}} = \left(\sum_{k=0}^{\infty} (2^{k(r-\alpha)}) ||\phi_k \times J^\alpha f||_{L_p}^q \right)^{1/q}.$$

Let $\{\psi_k\}_{k=0}^{\infty}$ be as in the proof of lemma 10.2. Then, by Young's inequality,

$$\begin{aligned} ||\phi_k \times J^\alpha f||_{L_p} &= ||\mathcal{F}^{-1}(\hat{\phi}_k \cdot \mathcal{F}(J^\alpha f))||_{L_p} = \\ &= ||\mathcal{F}^{-1}(\hat{\psi}_k (1+||\vec{\xi}||^2)^{\alpha/2} \hat{\phi}_k \hat{f})||_{L_p} = \\ &= ||\mathcal{F}^{-1}(\hat{\psi}_k (1+||\vec{\xi}||^2)^{\alpha/2}) \times \phi_k \times f||_{L_p} \leq \\ &\leq ||\mathcal{F}^{-1}(\hat{\psi}_k (1+||\vec{\xi}||^2)^{\alpha/2})||_{L_1} \cdot ||\phi_k \times f||_{L_p}. \end{aligned}$$

We will prove that here the first factor is finite. To fix the ideas, let $k > 0$. The case $k = 0$ is treated in the same way. Since $\hat{\psi}_k(\vec{\xi}) = \hat{\psi}(\vec{\xi}/2^k)$, we have

$$\begin{aligned} ||\mathcal{F}^{-1}(\hat{\psi}_k(\vec{\xi}) (1+||\vec{\xi}||^2)^{\alpha/2})||_{L_1} &= \\ &= ||\mathcal{F}^{-1}(\hat{\psi}(\vec{\xi}) (1+2^{2k}||\vec{\xi}||^2)^{\alpha/2})||_{L_1}. \end{aligned}$$

Put

$$g(\vec{\xi}) = \hat{\psi}(\vec{\xi}) (1+2^{2k}||\vec{\xi}||^2)^{\alpha/2}.$$

Then, by Schwarz' inequality and Parseval's formula,

$$\begin{aligned} ||\mathcal{F}^{-1}g||_{L_1} &\leq \left(\int (1+||\vec{x}||^{2N})^{-1} dx \right)^{1/2} \left(\int (1+||\vec{x}||^{2N})^{1/2} |\mathcal{F}^{-1}g|^2 dx \right)^{1/2} \leq \\ &\leq C (||g||_{L_2} + \sum_{\beta_1+\beta_2+\dots+\beta_N=2N} ||\vec{D}^{\vec{\beta}} g||_{L_2}). \end{aligned}$$

Here $\vec{D} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_N})$, $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_N)$. From this inequality, with our choice of g , we obtain

$$||\mathcal{F}^{-1}g||_{L_1} \leq C \cdot 2^{\alpha k}.$$

Thus

$$||\phi_k \times J^{\alpha} f||_{L_p} \leq C \cdot 2^{\alpha k}.$$

Using this in (10.24), the proof is concluded. $\#\#$

We have the following connection between Besov and Sobolev spaces:

Proposition 10.3. With continuous imbeddings we have

$$S_p^{\vec{r}1} B(A) \subset S_p^{\vec{r}} L(A) \subset S_p^{\vec{r}\infty} B(A).$$

(Here, of course, all spaces are connected with the same $\vec{N}$ in def. 10.4, 10.6).

Proof: It suffices to consider the case $d = 1$. By means of lemma 10.3, we can reduce to the case $r = 0$. But then the assertion is obvious:

$$S_p^{01} B(A) \subset S_p^0 L(A) \subset S_p^{0\infty} B(A). \quad \#\#$$

We are now ready for some interpolation results.

Theorem 10.3. Let $\vec{\theta} \in H_+^n$, $\vec{r}^i \in R^d$, $1 \leq p_i$, $p, q \leq \infty$ ($i = 0, 1, \dots, n$) and let $\vec{r} = \sum_{i=0}^n \theta_i \vec{r}^i$.

i) If R^d is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$(10.25) \quad (S_p^{\vec{r}^0} L(A), S_p^{\vec{r}^1} L(A), \dots, S_p^{\vec{r}^n} L(A))_{\vec{\theta}q} = S_p^{\vec{r}q} B(A).$$

ii) Regardless of $\{\vec{r}^i\}_0^n$, we have

$$(10.26) \quad S_p^{\vec{r}1} B(A) \subset (S_p^{\vec{r}^0} L(A), S_p^{\vec{r}^1} L(A), \dots, S_p^{\vec{r}^n} L(A))_{\vec{\theta}q;K} \subset S_p^{\vec{r}\infty} B(A),$$

$$(10.27) \quad S_p^{\vec{r}1} B(A) \subset (S_p^{\vec{r}^0} L(A), S_p^{\vec{r}^1} L(A), \dots, S_p^{\vec{r}^n} L(A))_{\vec{\theta}q;J} \subset S_p^{\vec{r}\infty} B(A).$$

iii) If $1/q = \sum_{i=0}^n \theta_i/p_i$, then

$$(10.28) \quad (S_{p_0}^{\vec{r}} L(A_0), S_{p_1}^{\vec{r}} L(A_1), \dots, S_{p_n}^{\vec{r}} L(A_n))_{\vec{\theta}q;K} = S_q^{\vec{r}} L(\vec{A}_{\vec{\theta}q;K}),$$

$$(10.29) \quad (S_{p_0}^{\vec{r}} L(A_0), S_{p_1}^{\vec{r}} L(A_1), \dots, S_{p_n}^{\vec{r}} L(A_n))_{\vec{\theta}q;J} = S_q^{\vec{r}} L(\vec{A}_{\vec{\theta}q;J}).$$

Proof: We choose vectors $\vec{u}^j \in R^d$ ($j = 0, 1, \dots, d$) in such manner that R^d is affinely spanned by $\{\vec{u}^j\}_0^d$ and $\vec{r}^i$ ($i = 0, 1, \dots, n$) belong to the open convex hull of $\{\vec{u}^j\}_0^d$. Then, by (10.17), for some $\vec{\theta}^i$ holds

$$(10.30) \quad (S_p^{\vec{u}^0} L, S_p^{\vec{u}^1} L, \dots, S_p^{\vec{u}^d} L)_{\vec{\theta}^i} = S_p^{\vec{r}^i} L \quad (i = 0, 1, \dots, n).$$

Thus, in view of prop. 10.3 and prop. 9.1, $S_p^{\vec{r}^i} L$ is of class $\mathcal{C}(\vec{\theta}^i; \{S_p^{\vec{u}^0} L, S_p^{\vec{u}^1} L, \dots, S_p^{\vec{u}^d} L\})$.

Under the assumptions of i), R^{d+1} is (linearly) spanned by $\{\vec{\theta}^i\}_0^n$. Moreover, condition $\mathcal{F}$ is satisfied, by prop. 10.2.

Application of th. 9.1 then yields

$$(S_p^{\vec{r}^0} L, S_p^{\vec{r}^1} L, \dots, S_p^{\vec{r}^n} L)_{\vec{\theta}^q} = (S_p^{\vec{u}^0} L, S_p^{\vec{u}^1} L, \dots, S_p^{\vec{u}^d} L)_{\vec{\lambda}^q},$$

where $\sum_{i=0}^n \theta_i \vec{r}^i = \sum_{i=0}^d \lambda_i \vec{u}^i = \vec{r}$. But here, by (10.17),

$$(S_p^{\vec{u}^0} L, S_p^{\vec{u}^1} L, \dots, S_p^{\vec{u}^d} L)_{\vec{\lambda}^q} = S_p^{\vec{r}^q} L.$$

This proves i). Part ii) also follows from (10.30), now by means of prop. 9.6. In iii), finally, by means of lemma 10.3 we can reduce to the case $\vec{r} = 0$. But then (10.28) and (10.29) follows directly from th. 8.1 and th. 8.3 respectively. $\#$

Concerning condition $\mathcal{F}$, we have:

Proposition 10.4. Let $\vec{r} \in R^d$, $\vec{r}^i \in R^d$ ($i = 0, 1, \dots, n$).

i) If R^d is affinely spanned by $\{\vec{r}^i\}_0^n$, then

$$\{S_p^{\vec{r}^0} L(A), S_p^{\vec{r}^1} L(A), \dots, S_p^{\vec{r}^n} L(A)\}$$

satisfies the condition $\mathcal{F}$.

ii) If $\vec{A} = \{A_0, A_1, \dots, A_n\}$ satisfies the condition $\mathcal{F}$,

so does also

$$\{S_p^{\vec{r}} L(A_0), S_p^{\vec{r}} L(A_1), \dots, S_p^{\vec{r}} L(A_n)\}.$$

Proof: By means of prop. 9.3, part i) follows from the proof of i) in th. 10.3. Part ii) is a consequence of prop. 8.1, in view of the isomorphism (10.22). $\neq$

10.4. Some imbedding relations.

For the sake of completeness we will list, without proofs, some imbedding properties of the spaces $S_p^{\vec{r}q}B$ and $S_p^{\vec{r}}L$. Actually, considering vector valued functions, we need only refer to the corresponding results in the case $d = 1$. For the proofs in that case, see Nikolskij [14] or Besov [3], cf. also Peetre [19] for a treatment using interpolation. In the case of general d , for "ad hoc" proofs, see e.g. Nikolskij [15], Amanov [1].

We consider functions, defined on $R^{\vec{N}}$, $\vec{N} = (N_1, N_2, \dots, N_d)$, in the sense of def. 10.4 and def. 10.6. With continuous imbeddings we have:

$$\begin{aligned} S_p^{\vec{r}q}B &\subset S_p^{\vec{r}'q'}B && \text{if } q \leq q', \\ S_p^{\vec{r}q}B &\subset S_p^{\vec{r}'q}B && \text{if } \vec{r}' \leq \vec{r}, \\ S_p^{\vec{r}q}B &\subset S_{p'}^{\vec{r}'q}B && \text{if } \vec{r} - \vec{N}/p = \vec{r}' - \vec{N}/p', \quad p \leq p', \end{aligned}$$

and

$$\begin{aligned} S_p^{\vec{r}}L &\subset S_p^{\vec{r}'}L && \text{if } \vec{r}' \leq \vec{r}, \\ S_p^{\vec{r}}L &\subset S_{p'}^{\vec{r}'}L && \text{if } \vec{r} - \vec{N}/p = \vec{r}' - \vec{N}/p', \quad p \leq p' \text{ and} \\ &&& 1 < p < \infty. \end{aligned}$$

We also remind of (prop. 10.3)

$$S_p^{\vec{r}1}B \subset S_p^{\vec{r}}L \subset S_p^{\vec{r}\infty}B.$$

Appendix. A modified fundamental lemma.

In Section 5 we studied the equivalence between K- and J-spaces. To obtain a sufficient condition for equivalence, we introduced condition $\mathcal{F}$ (def. 5.2), which in the case $n = 1$ reduces to the fundamental lemma. We also saw that for $n > 1$ there exist $(n+1)$ -tuples, which do not satisfy condition $\mathcal{F}$.

In this appendix, in th. A.1 we give another condition, which in the case $n = 1$ reduces to the fundamental lemma. For $n > 1$ this condition is satisfied by all $(n+1)$ -tuples.

We also define an interpolation functor (the I-functor, def. A.2) to which th. A.1 applies. However, this interpolation functor has great defects and seems to be of little value compared to the K- and J-functors.

Let $\vec{A} = \{A_0, A_1, \dots, A_n\}$ be a Banach $(n+1)$ -tuple. We shall need the following functional:

Defintion A.1. Set

$$E_j = \{\vec{t} \in R_+^{n+1} \mid \max(\vec{t}) = t_j\} \quad (j = 0, 1, \dots, n).$$

Let $a \in \Sigma(\vec{A})$. For fixed j ($j = 0, 1, \dots, n$) we define (admitting infinite values)

$$\begin{aligned} I(\vec{t}, a; \vec{A}) &= \|a\|_{t_j A_j \cap (t_0 A_0 + \dots + t_j^{\vee} A_j + \dots + t_n A_n)} \\ &= \max(t_j \|a\|_{A_j}, K((t_0, \dots, t_j^{\vee}, \dots, t_n), a; \{A_0, \dots, A_j^{\vee}, \dots, A_n\})) \\ &\text{if } \vec{t} \in E_j. \end{aligned}$$

We then have (cf. prop. 2.1)

Proposition A.1.

$$\begin{aligned} (A.1) \quad K(\vec{s}, a; \vec{A}) &\leq \min(s_j/t_j, \max_{i \neq j} s_i/t_i) I(\vec{t}, a; \vec{A}) \\ &\text{if } \vec{t} \in E_j \quad (j = 0, 1, \dots, n), \end{aligned}$$

$$(A.2) \quad I(\vec{t}, a; \vec{A}) \leq \max(t_j/s_j, \min_{i \neq j} t_i/s_i) J(\vec{s}, a; \vec{A})$$

if $\vec{t} \in E_j \quad (j = 0, 1, \dots, n)$.

In particular

$$(A.1') \quad \|a\|_{\Sigma(\vec{A})} \leq \min(1/\vec{t}) I(\vec{t}, a; \vec{A}),$$

$$(A.2') \quad I(\vec{t}, a; \vec{A}) \leq \max(\vec{t}) \|a\|_{\Delta(\vec{A})}.$$

Proof: Let $\vec{t} \in E_j$. Then

$$\begin{aligned} K(\vec{s}, a; \vec{A}) &\leq \min(s_j \|a\|_{A_j}, K((s_0, \dots, \check{s}_j, \dots, s_n), a; \{A_0, \dots, \check{A}_j, \dots, A_n\})) \leq \\ &\leq \min(\frac{s_j}{t_j} t_j \|a\|_{A_j}, \max_{i \neq j}(\frac{s_i}{t_i}) K((t_0, \dots, \check{t}_j, \dots, t_n), a; \{A_0, \dots, \check{A}_j, \dots, A_n\})) \leq \\ &\leq \min(\frac{s_j}{t_j}, \max_{i \neq j} \frac{s_i}{t_i}) I(\vec{t}, a; \vec{A}). \end{aligned}$$

(A.2) is proved in a similar way. $\neq$

If $n = 1$, then $I(\vec{t}, a) = J(\vec{t}, a)$. In that case, the following theorem reduces to the fundamental lemma.

Theorem A.1. If

$$(A.3) \quad K(\vec{t}, a; \vec{A}) \rightarrow 0 \quad \text{as} \quad t_i \rightarrow 0, \quad \max(\vec{t}) = 1 \quad (i = 0, 1, \dots, n),$$

then there exists a function $u \in \mathcal{H}_0(\Sigma(\vec{A}))$ such that

$$(A.4) \quad I(\vec{t}, u(\vec{t}); \vec{A}) \leq C K(\vec{t}, a; \vec{A})$$

and

$$(A.5) \quad a = \int u(\vec{t}) \omega(\vec{t})$$

with convergence in $\Sigma(\vec{A})$ in the sense of a generalized Lebesgue integral. Here C is independent of $\vec{t}, a$ and $\vec{A}$. If a obeys the stronger condition

$$(A.3') \quad \int \frac{K(\vec{t}, a; \vec{A})}{\max(\vec{t})} \omega(\vec{t}) < \infty$$

(i.e. $a \in \sigma(\vec{A})$, cf. def. 5.1) then the integral in (A.5) converges absolutely in $\Sigma(\vec{A})$.

(Note that, by (A.4), $u(\vec{t}) \in A_j \cap (A_0 + \dots + \check{A}_j + \dots + A_n)$ a.e. in E_j ($j = 0, 1, \dots, n$)).

Proof: We can find $a_j(\vec{t}) \in \mathcal{H}_0(A_j)$ ($j = 0, 1, \dots, n$) such that

$$\sum_{j=0}^n a_j(\vec{t}) = a,$$

$$\sum_{j=0}^n t_j ||a_j(\vec{t})||_{A_j} \leq 2K(\vec{t}, a).$$

Let ϕ be a non-negative 0-homogeneous C^∞ -function on R_+^{n+1} , vanishing outside some cone with vertex in the origin. Suppose that

$$\int \phi(\vec{t}) \omega(\vec{t}) = 1.$$

(Considered as a function on P_+^n , ϕ thus has compact support).

We define

$$\alpha_j(\vec{t}) = \int \phi(\vec{t}/\vec{s}) a_j(\vec{s}) \omega(\vec{s}) \quad (j = 0, 1, \dots, n).$$

Then α_j is differentiable of any order and

$$(A.6) \quad \sum_{j=0}^n \alpha_j(\vec{t}) = a.$$

Moreover $\alpha_j \in \mathcal{H}_0(A_j)$ ($j = 0, 1, \dots, n$), which follows from

$$\begin{aligned} t_j ||\alpha_j(\vec{t})||_{A_j} &\leq \int t_j / s_j \phi(\vec{t}/\vec{s}) s_j ||a_j(\vec{s})||_{A_j} \omega(\vec{s}) \leq \\ &\leq \int t_j / s_j \phi(\vec{t}/\vec{s}) K(\vec{s}, a) \omega(\vec{s}) \leq \\ &\leq \int t_j / s_j \max(\vec{s}/\vec{t}) \phi(\vec{t}/\vec{s}) K(\vec{t}, a) \omega(\vec{s}) = CK(\vec{t}, a). \end{aligned}$$

Here $C = \int 1/s_j \max(\vec{s}) \phi(1/\vec{s}) \omega(\vec{s})$. Combined with the assumption (A.3), this estimate also yields

$$(A.7) \quad ||\alpha_j(\vec{t})||_{A_j} \rightarrow 0 \quad \text{as} \quad \max(\vec{t}) = t_j = 1, \quad t_i \rightarrow 0$$

for some $i \neq j$ ($i, j = 0, 1, \dots, n$).

Now set

$$\partial_j = (t_0 \frac{\partial}{\partial t_0}) \dots (t_j \frac{\partial}{\partial t_j}) \dots (t_n \frac{\partial}{\partial t_n}) \quad (j = 0, 1, \dots, n).$$

We define

$$u(\vec{t}) = \partial_j \alpha_j(\vec{t}) \quad \text{if} \quad \vec{t} \in E_j \quad (j = 0, 1, \dots, n).$$

Obviously $u \in \mathcal{H}_0(\Sigma(\vec{A}))$. In view of (A.7) we have

$$\int_{\max(\vec{t})=t_j=1} u(\vec{t}) \omega(\vec{t}) = \int_{\max(\vec{t})=t_j=1} \frac{\partial}{\partial t_0} \dots \frac{\partial}{\partial t_j} \dots \frac{\partial}{\partial t_n} \alpha_j(\vec{t}) dt_0 \dots dt_j \dots dt_n = \alpha_j(\vec{t}_n)$$

in the sense of a generalized Lebesgue integral. Hence, in the same sense,

$$\int u(\vec{t}) \omega(\vec{t}) = \sum_{j=0}^n \alpha_j(\vec{t}_n) = a.$$

By this (A.5) is verified.

To verify (A.4), we first note that, in view of (A.6),

$$(A.8) \quad u(\vec{t}) = \partial_j \alpha_j(\vec{t}) = - \sum_{\substack{i=0 \\ i \neq j}}^n \partial_j \alpha_i(\vec{t}) \quad (j = 0, 1, \dots, n).$$

Set

$$\psi_j(\vec{t}) = \frac{\partial}{\partial t_0} \dots \frac{\partial}{\partial t_j} \dots \frac{\partial}{\partial t_n} \phi(\vec{t}).$$

With ∂_j acting on the variable $\vec{t}$, we have

$$\begin{aligned} t_i ||\partial_j \alpha_i(\vec{t})||_{A_i} &\leq t_i \int ||\partial_j \phi(\vec{t}/\vec{s})|| ||a_i(\vec{s})||_{A_i} \omega(\vec{s}) = \\ &= \int \frac{t_0}{s_0} \dots \frac{t_j}{s_j} \dots \frac{t_n}{s_n} \frac{t_i}{s_i} |\psi_j(\vec{t}/\vec{s})| s_i ||a_i(\vec{s})||_{A_i} \omega(\vec{s}) \leq \\ &\leq \int \frac{t_0}{s_0} \dots \frac{t_j}{s_j} \dots \frac{t_n}{s_n} \frac{t_i}{s_i} |\psi_j(\vec{t}/\vec{s})| K(\vec{s}, a) \omega(\vec{s}) \leq \\ &\leq \int \frac{t_0}{s_0} \dots \frac{t_j}{s_j} \dots \frac{t_n}{s_n} \frac{t_i}{s_i} |\psi_j(\vec{t}/\vec{s})| \max(\vec{s}/\vec{t}) K(\vec{t}, a) \omega(\vec{s}) = CK(\vec{t}, a). \end{aligned}$$

Here $C = \int s_j/s_i |1/\vec{s}| |\psi_j(1/\vec{s})| \max(\vec{s}) \omega(\vec{s})$. Thus, in view of (A.8), for some constant C holds

$$||u(\vec{t})||_{t_j A_j} = t_j ||\partial_j \alpha_j(\vec{t})||_{A_j} \leq CK(\vec{t}, a),$$

$$||u(\vec{t})||_{t_0 A_0 + \dots + t_j A_j + \dots + t_n A_n} \leq \sum_{\substack{i=0 \\ i \neq j}}^n t_i ||\partial_j \alpha_i(\vec{t})||_{A_i} \leq \\ \leq CK(\vec{t}, a).$$

This proves (A.4).

The last assertion, about the absolute convergence, is proved by means of prop. A.1 (cf. also rem. 5.3). Namely, in view of (A.1') and (A.4) we have

$$||u(\vec{t})||_{\Sigma(\vec{A})} \leq \frac{I(\vec{t}, u(\vec{t}))}{\max(\vec{t})} \leq C \frac{K(\vec{t}, a)}{\max(\vec{t})}.$$

Thus, if a obeys (A.3'), then

$$\int ||u(t)||_{\Sigma(\vec{A})} \omega(\vec{t}) < \infty.$$

This completes the proof. $\#$

It now seems worth-while to make

Definition A.2. Let $\vec{\theta} \in H_+^n$, $1 \leq q \leq \infty$. By $\vec{A}_{\vec{\theta}q; I}$ we mean

the set of all $a \in \Sigma(\vec{A})$ for which there exists a function $u \in \mathcal{H}_0(\Sigma(\vec{A}))$ such that

$$(A.9) \quad a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}),$$

$$(A.10) \quad \phi_{\vec{\theta}q} [I(\vec{t}, u(\vec{t}); \vec{A})] < \infty.$$

Note that, by (A.10), $u(\vec{t}) \in A_j \cap (A_0 + \dots + \check{A}_j + \dots + A_n)$ a.e. in E_j ($j = 0, 1, \dots, n$). Concerning the convergence of the integral in (A.9), cf. remark A.1 below. We also observe that if $n = 1$, then $\vec{A}_{\vec{\theta}q; I} = \vec{A}_{\vec{\theta}q; J}$.

Def. A.1 and A.2 have several defects compared to the K- and J-analogues. We mention a few. For the K- and J-functionals we have the relations

$$K(\vec{t}, a; \vec{s}\vec{A}) = K(\vec{t}\vec{s}, a; \vec{A}),$$

$$J(\vec{t}, a; \vec{s}\vec{A}) = J(\vec{t}\vec{s}, a; \vec{A}).$$

But for the I-functional we have in general

$$I(\vec{t}, a; \vec{s}\vec{A}) \neq I(\vec{t}\vec{s}, a; \vec{A}).$$

This is caused by the fact, that in def. A.1 the point $(1, 1, \dots, 1)$ is privileged. Furthermore, in def. A.2 we observe that if $u(\vec{t})$ obeys (A.9), then in general $u(\vec{s}\vec{t})$ does not. We recall that in the J-case it did, and that this was of great value on several places.

In spite of these objections we precede one step further:

Theorem A.2. $\vec{A}_{\vec{\theta}q;I}$ is a Banach space under the norm

$$a \mapsto \|a\|_{\vec{A}_{\vec{\theta}q;I}} = \inf_{\vec{\theta}q} \Phi_{\vec{\theta}q} [I(\vec{t}, u(\vec{t}); \vec{A})].$$

Here the infimum is taken over all functions u obeying (A.9) and (A.10). Furthermore

$$(A.11) \quad K(\vec{t}, a; \vec{A}) \leq C \max(\vec{t}) \|a\|_{\vec{A}_{\vec{\theta}q;I}} \quad \text{if } a \in \vec{A}_{\vec{\theta}q;I},$$

$$(A.12) \quad \|a\|_{\vec{A}_{\vec{\theta}q;I}} \leq C \max(1/\vec{t}) J(\vec{t}, a; \vec{A}) \quad \text{if } a \in \Delta(\vec{A}).$$

Hence

$$(A.13) \quad \Delta(\vec{A}) \subset \vec{A}_{\vec{\theta}q;I} \subset \Sigma(\vec{A}).$$

The functor

$$\mathcal{C}_n \ni \vec{A} \mapsto \vec{A}_{\vec{\theta}q;I} \in \mathcal{C}_0,$$

referred to as the I-functor, is an exact interpolation functor.

Proof: We begin with (A.11). Once this is verified, the only non-trivial norm condition, that $\|a\|_{\vec{A}_{\vec{\theta}q;I}} = 0$ implies $a = 0$, will follow.

In view of prop. 2.1 it suffices to show

$$||a||_{\Sigma(\vec{A})} \leq C ||a||_{\vec{A}_{\vec{\theta}_q; I}}.$$

Let u obey (A.9). Then, in view of (A.1') and Hölder's inequality,

$$\begin{aligned} (A.14) \quad \int_{\max(\vec{t})=1} ||u(\vec{t})||_{\Sigma(\vec{A})} \omega(\vec{t}) &\leq \int_{\max(\vec{t})=1} I(\vec{t}, u(\vec{t})) \omega(\vec{t}) \leq \\ &\leq \left(\int_{\max(\vec{t})=1} |\vec{t}^{\vec{\theta}}|^{q'} \omega(\vec{t}) \right)^{1/q'} \phi_{\vec{\theta}_q} [I(\vec{t}, u(\vec{t}))], \end{aligned}$$

where $1/q + 1/q' = 1$. Hence

$$||a||_{\Sigma(\vec{A})} \leq \int ||u(\vec{t})||_{\Sigma(\vec{A})} \omega(\vec{t}) \leq C \phi_{\vec{\theta}_q} [I(\vec{t}, u(\vec{t}))].$$

Making u vary, we obtain (A.11).

Turning to (A.12), in view of prop. 2.1 it suffices to show

$$||a||_{\vec{A}_{\vec{\theta}_q; I}} \leq C ||a||_{\Delta(\vec{A})}.$$

To this end, let ϕ be a non-negative 0-homogeneous function on R_+^{n+1} , vanishing outside some cone with vertex in the origin.

Let $a \in \Delta(\vec{A})$. We define

$$u(\vec{t}) = \phi(\vec{t})a.$$

Then, by (A.2'),

$$I(\vec{t}, u(\vec{t})) \leq \max(\vec{t}) \phi(\vec{t}) ||a||_{\Delta(\vec{A})}.$$

Thus

$$\begin{aligned} ||a||_{\vec{A}_{\vec{\theta}_q; I}} &\leq \phi_{\vec{\theta}_q} [I(\vec{t}, u(\vec{t}))] \leq \\ &\leq \left(\int |\vec{t}^{-\vec{\theta}}| \max(\vec{t}) \phi(\vec{t})^q \omega(\vec{t}) \right)^{1/q} ||a||_{\Delta(\vec{A})}. \end{aligned}$$

This settles (A.12).

To verify the completeness of $\vec{A}_{\vec{\theta}q;I}$, let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\vec{A}_{\vec{\theta}q;I}$ such that

$$\sum_{n=1}^{\infty} \|a_n\|_{\vec{A}_{\vec{\theta}q;I}} < \infty.$$

Then one can find functions $u_n \in \mathcal{H}_0(\Sigma(\vec{A}))$ such that

$$a_n = \int u_n(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A})$$

and

$$\sum_{n=1}^{\infty} \Phi_{\vec{\theta}q} [I(\vec{t}, u_n(\vec{t}))] < \infty.$$

By the last condition, $\sum_{n=1}^{\infty} u_n(\vec{t})$ converges a.e.. Let $u(\vec{t})$ be its sum. Obviously $u \in \mathcal{H}_0(\Sigma(\vec{A}))$. Since

$$I(\vec{t}, u(\vec{t})) \leq \sum_{n=1}^{\infty} I(\vec{t}, u_n(\vec{t})),$$

we have

$$(A.15) \quad \Phi_{\vec{\theta}q} [I(\vec{t}, u(\vec{t}))] \leq \sum_{n=1}^{\infty} \Phi_{\vec{\theta}q} [I(\vec{t}, u_n(\vec{t}))] < \infty.$$

Thus, by (A.14), $\int u(\vec{t}) \omega(\vec{t})$ is absolute convergent in $\Sigma(\vec{A})$.

Let its value be a . From (A.15) then follows that $a \in \vec{A}_{\vec{\theta}q;I}$ and $\sum_{n=1}^{\infty} a_n = a$. This proves the completeness of $\vec{A}_{\vec{\theta}q;I}$.

Finally, we consider the interpolation property. Let $T : \vec{A} \rightarrow \vec{B}$ and let $\vec{M} = \vec{M}(T)$ be as in def. 1.3. Let $a \in \vec{A}_{\vec{\theta}q;I}$, $u \in \mathcal{H}_0(\Sigma(\vec{A}))$ with

$$a = \int u(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{A}).$$

Set

$$v(\vec{t}) = Tu(\vec{t}).$$

Then $v \in \mathcal{H}_0(\Sigma(\vec{B}))$,

$$Ta = \int v(\vec{t}) \omega(\vec{t}) \quad \text{in } \Sigma(\vec{B}),$$

and

$$I(\vec{t}, v(\vec{t}); B) \leq I(\vec{t}, u(\vec{t}); \vec{A}) \leq \max(\vec{M}) I(\vec{t}, u(\vec{t}); \vec{A}).$$

By application of $\Phi_{\vec{\theta}_q}$ and variation of u we then obtain

$$\|Ta\|_{\vec{B}_{\vec{\theta}_q; I}} \leq \max(\vec{M}) \|a\|_{\vec{A}_{\vec{\theta}_q; I}}.$$

Together with (A.13), this proves the exactness of the I-functor. $\#$

Remark A.1. As a consequence of (A.14), the absolute convergence of the integral in (A.9) follows from (A.10). (Cf. remark 4.3 for the analogous property of the J-spaces).

Theorem A.3. We have

$$\vec{A}_{\vec{\theta}_q; J} \subset \vec{A}_{\vec{\theta}_q; K} \subset \vec{A}_{\vec{\theta}_q; I}.$$

Proof: The first inclusion is a restatement of prop. 5.1. Since $\vec{A}_{\vec{\theta}_q; K} \in \sigma(\vec{A})$ (cf. remark 5.2), the second one is a consequence of th. A.1. $\#$

We conclude by deriving the I-spaces explicitly in two simple cases.

Example 1. Consider the triple $\vec{A} = \{A_0, A_1, A_2\}$, where $A_0 \subset A_1 = A_2$, $\|a\|_{A_1} = \|a\|_{A_2} \leq \|a\|_{A_0}$ if $a \in A_0$. Then, in def. A.2, for almost every $\vec{t}$ we have

$$\begin{aligned} u(\vec{t}) &\in A_0 \cap A_1 & \text{if } \vec{t} \in E_0, \\ u(\vec{t}) &\in A_1 & \text{if } \vec{t} \in E_1 \cup E_2. \end{aligned}$$

It is readily seen that

$$I(\vec{t}, u(\vec{t})) = t_i \|u(\vec{t})\|_{A_i} \quad \text{if } \vec{t} \in E_i \quad (i = 0, 1, 2).$$

Let $a \in A_1$ and let ϕ be a non-negative 0-homogeneous function on R_+^3 , vanishing outside some cone in $E_1 \cup E_2$ with vertex in the origin. Suppose that

$$\int \phi(\vec{t}) \omega(\vec{t}) = 1.$$

Set

$$u(\vec{t}) = \phi(\vec{t})a.$$

Then

$$a = \int u(\vec{t})\omega(\vec{t}) \quad \text{in} \quad \Sigma(\vec{A})$$

and

$$\begin{aligned} \|a\|_{\vec{A}_{\vec{\theta}_q; I}} &\leq \Phi_{\vec{\theta}_q} [I(\vec{t}, u(\vec{t}))] = \\ &= \left(\int_{E_1 \cup E_2} (|\vec{t} - \vec{\theta}_1|_{\max(t_1, t_2)} \phi(\vec{t}))^q \omega(\vec{t}) \right)^{1/q} \|a\|_{A_1} < \infty. \end{aligned}$$

Hence, at least algebraically, $\vec{A}_{\vec{\theta}_q; I} = A_1$. By prop. 6.4, $\vec{A}$ satisfies the condition $\mathcal{F}$ and thus $\vec{A}_{\vec{\theta}_q; J} = \vec{A}_{\vec{\theta}_q; K}$. But by appropriate choices of A_0 and A_1 one easily attains $\vec{A}_{\vec{\theta}_q; J} = \vec{A}_{\vec{\theta}_q; K} \neq A_1 = \vec{A}_{\vec{\theta}_q; I}$. This example shows that the condition $\mathcal{F}$ does not guarantee equivalence between the I-spaces and the K- and J-spaces.

Example 2. In the previous example we had $\vec{A}_{\vec{\theta}_q; I} = \Sigma(\vec{A})$.

The I-functor is not trivial in the sense that so is always the case. To see this, consider the triple $\vec{A} = \{A_0, A_1, A_2\}$, where $A_0 = A_1 \subset A_2$, $\|a\|_{A_2} \leq \|a\|_{A_0} = \|a\|_{A_1}$ if $a \in A_0 = A_1$. Then, in def. A.2, we have $u(\vec{t}) \in A_0$ a.e. In particular, let $A_2 = \ell^{01}$, i.e. the Banach space of real sequences $\{a_n\}_{n=0}^{\infty}$ corresponding to the norm $\sum_{n=0}^{\infty} |a_n|$, and let $A_0 = A_1$ be the subspace determined by the condition $a_0 = 0$. Then $\vec{A}_{\vec{\theta}_q; I} = A_0 \neq A_1 = \Sigma(\vec{A})$. Moreover, in this case clearly holds $\vec{A}_{\vec{\theta}_q; J} = \vec{A}_{\vec{\theta}_q; K} = \vec{A}_{\vec{\theta}_q; I}$.

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